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VALERIJONAS DUMSKIS

ANALYSIS AND APPLICATION OF METHODS FOR SEARCH OF STOCHASTIC EQUILIBRIUM

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Scientific Supervisor

Prof. Dr. Habil. Leonidas Sakalauskas (Vilnius University, Technological Sciences, Informatics Engineering – 07 T).

The dissertation will be defended at the Council of the Scientific Field of Informatics Engineering at the Institute of Mathematics and Informatics of Vilnius University:

Chairman

Prof. Dr. Habil. Antanas Žilinskas (Vilnius University, Technological Sciences, Informatics Engineering – 07 T).

Members:

Prof. Dr. Habil. Juozas Augutis (Vytautas Magnus University, Physical Sciences, Informatics – 09 P),

Prof. Dr. Kęstutis Dučinskas (Klaipėda University, Physical Sciences, Informatics – 09 P),

Dr. Virginijus Marcinkevičius (Vilnius University, Technological Sciences, Informatics Engineering – 07 T),

Prof. Dr. Habil. Edmundas Kazimieras Zavadskas (Vilnius Gediminas Technical University, Technological Sciences, Informatics Engineering – 07 T).

Opponents:

Prof. Dr. Eduardas Bareiša (Kaunas University of Technology, Technological Sciences, Informatics Engineering – 07 T),

Dr. Audronė Jakaitienė (Vilnius University, Physical Sciences, Informatics – 09 P).

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Address: Akademijos st. 4, LT-08663 Vilnius, Lithuania.

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VILNIAUS UNIVERSITETAS

VALERIJONAS DUMSKIS

STOCHASTINĖS PUSIAUSVYROS PAIEŠKOS METODŲ TYRIMAS IR TAIKYMAS

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Mokslinis vadovas

prof. habil. dr. Leonidas Sakalauskas (Vilniaus universitetas, technologijos mokslai, informatikos inžinerija – 07 T).

Disertacija ginama Vilniaus universiteto Matematikos ir informatikos instituto Informatikos inžinerijos krypties taryboje:

Pirmininkas

prof. habil. dr. Antanas Žilinskas (Vilniaus universitetas, technologijos mokslai, informatikos inžinerija – 07 T).

Nariai:

prof. habil. dr. Juozas Augutis (Vytauto Didžiojo universitetas, fiziniai mokslai, informatika – 09 P),

prof. dr. Kęstutis Dučinskas (Klaipėdos universitetas, fiziniai mokslai, informatika – 09 P),

dr. Virginijus Marcinkevičius (Vilniaus universitetas, technologijos mokslai, informatikos inžinerija – 07 T),

prof. habil. dr. Edmundas Kazimieras Zavadskas (Vilniaus Gedimino technikos universitetas, technologijos mokslai, informatikos inžinerija – 07 T).

Oponentai:

prof. dr. Eduardas Bareiša (Kauno technologijos universitetas, technologijos mokslai, informatikos inžinerija – 07 T),

dr. Audronė Jakaitienė (Vilniaus universitetas, fiziniai mokslai, informatika – 09 P).

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1. Introduction

1.1. Research Area

The state of equilibrium and related issues are relevant in many areas of economics, business and financial management. Usually the equilibrium is an outcome of some dynamic process. It is important to identify in what conditions the equilibrium exists and, if it exists, if it is the only one, how it is obtained, what is the degree of stability of the equilibrium. The search for the equilibrium is of major importance in economic analysis applications: forecasting the outcome and evaluating proper values of parameters of the model, comparing results of the experiment with forecasts of the model, testing the designed mechanism. In applications the Nash equilibrium is often analysed, in which none of the competing individuals (players), who behave non-cooperatively, has an intention (it is useless for him) to change his strategy one-sidedly when other individuals follow the equilibrium strategy. If players make decisions hierarchically, we have *Stackelberg equilibrium*. There are many areas of economics and finances where one or another type of uncertainty is encountered; e.g., the demand depends on certain random values or some values cannot be accurately identified or measured. Then we have to analyse the stochastic equilibrium.

In practical problems, when we have many different players, we must explore the dynamics of their behaviour, identify marginal cases of this dynamics and find values of parameters, in the presence of which this dynamics leads to the equilibrium.

1.2. Relevance of the Problem

If the decision is made by several players whose interests do not coincide and they cannot cooperate, the game is non-cooperative, when the player, choosing from strategies available for him, seeks to maximize (minimize) the objective function, the value of which also depends on other players' strategies. In 1950, J. Nash proposed to apply the conception of the equilibrium for non-cooperative games. According to Nash, in the equilibrium none of players intends to change his strategy (in such case he would only get the same or less), if all other players follow equilibrium strategies.

A big share of equilibrium situations is modelled employing bilevel decision-making models, in which the order of decision-making is important. These models, referred to as *Stackelberg games*, in which the feasible set is determined by the set of

optimal solutions of the second, parametric optimization problem. Models of this type are applied in various areas, where at the top level the leader maximizes (minimizes) his objective function, which depends both on his strategy and on the follower's strategy, chooses the strategy, and the follower reacting to it makes a decision at the bottom level so as to maximize (minimize) his objective function.

Very often we have to evaluate both payoff and loss, which depend on random values and risk. Risk evaluation issues are very important in financial markets. Then it is necessary to include some risk measure into the model: this can be *value at risk* (VaR) or *conditional value at risk* (CVaR).

1.3. Object of Research

The research object of the dissertation is investigation of the model of heterogeneous agents and its application for modelling stochastic Nash and Stackelberg equilibriums by Monte Carlo method.

1.4. The Aim and Objectives of the Research

The research aim is to identify the impact of heterogeneous agents on the formation of the economic bubble, to create and examine algorithms for special bilevel stochastic programming problems and for search of the stochastic Nash equilibrium applying Monte Carlo method. To implement this aim, the following objectives are solved:

1. To design the mathematical model of the economic bubble and its crash and, employing it, to examine the Lithuanian real estate bubble.
2. To create the Monte Carlo algorithm to optimize conditional value risk in case if constraints contain conditional value at risk.
3. To investigate the method of importance sampling and apply it to solve a two stage stochastic linear programming problem.
4. To investigate the stochastic Nash equilibrium and create the algorithm for search of this equilibrium, to evaluate the behaviour of this algorithm.

1.5. Scientific Novelty

New results have been obtained in the research:

1. The mathematical model of the economic bubble and its crash, which includes agents of two different types, has been proposed.

2. The Monte Carlo algorithm for the solution of the stochastic programming problem the objective function and constraints of which contain the conditional value at risk has been created and investigated.
3. The algorithm of importance sampling, which is applied to solve the two stage stochastic linear programming problem, has been created and investigated.
4. The Monte Carlo algorithm for search of the stochastic Nash equilibrium has been created and investigated, applying statistic criteria for its stopping..

1.6. Practical Significance of the Results

The following practical results have been obtained in the research:

1. The mathematical model of unsustainable status (of the bubble and its crash) has been proposed and it has been employed for investigation of the real estate bubble of Lithuania.
2. The Monte Carlo algorithm has been created to optimize the conditional value at risk when conditional value at risk is in constraints; this algorithm has been investigated, solving generated test problems.
3. The importance sampling algorithm for two stage stochastic linear programming has been created and investigated solving the test problem.
4. The algorithm for search of stochastic Nash equilibrium was created and applied to solve the problem of electricity supply with initial contracts.

1.7. Defended Statements

1. Having included agents of two different kinds into the mathematical model of the economic bubble, we obtain a more accurate estimation of the beginning of the bubble and its crash.
2. Monte Carlo algorithms for sequential search for stochastic Nash or Stackelberg equilibrium are characterized by good convergence and enable to find the equilibrium strategy at proper accuracy, terminating the algorithms according to a statistical criterion.

1.8. Approbation and Publications of the Research

The results of the dissertation were presented at international (3 papers) and national (4 papers) scientific conferences. Research results were announced in scientific publications and in reviewed recognised international data bases (the list approved by the

Research Council of Lithuania): 2 articles and 1 article accepted for printing in scientific journals which are included in *CEEOL* and *Index Copernicus*, 1 article in *MatSciNet* (*Mathematical Reviews*), 1 publication in International conference proceedings, included in the list of the Institute of Scientific Information, 1 publication in reviewed Lithuanian international conference proceedings.

1.9. Structure of the Dissertation

The dissertation consists of five chapters, the list of references and the appendix.

Chapter 1 is introduction. **Chapter 2** presents formulation of problems for search of the equilibrium: the model of heterogeneous agents, the stochastic bilevel programming problem, the model of the Nash equilibrium, and their analytical investigation. **Chapter 3** presents the investigation of financial bubbles of the market and their crashes: using a Ponzi scheme, the proposed mathematical model for financial bubbles and their crashes, and the model of heterogeneous agents. **Chapter 4** presents the investigation of stochastic bilevel programming problems (the problem where the objective function and constraints contain a conditional value at risk, solving of a two stage stochastic linear programming problem, employing the method of importance sampling).**Chapter 5** contains the investigation of the algorithm for search of the stochastic Nash equilibrium and its application.

At the end of the dissertation conclusions, the list of references and the appendix are presented.

2. The Problems of Equilibrium Search

2.1. Introduction

Phenomena of stochastic equilibrium arise in many fields of economy and finance usually analyzed as stochastic Nash and/or Stackelberg equilibrium. Multiple decision-making is often related with competition of several decision makers/players seeking one's own interests. Very often these players are non-cooperative and their decisions intervene with each other, what calls forth the problem of finding Nash equilibrium.

The manager of the distribution network, the supplier in the communication market, the supplier in a supply chain acts as a leader and makes a decision first. Then followers–the users of these networks, retailers choose their strategy according to the decision which is made by the leader. In the game theory and in the multilevel programming the hierarchical decision-making problems are analyzed and it is aimed to find Stackelberg (equilibrium) solutions for these problems.

M. Patriksson and L. Wynter (1997) maintain, uncertainty occurs almost in all applications of hierarchical problems and ignoring or simplifying it leads to an expensive mistake. Different authors in the different ways include the uncertainty in their models. For example, L. Cheng et al. analyse the model of the bilevel formation of the prices and orders between producer and retailer where demand is uniformly distributed (2009). In this model they include CVaR as retailer's objective function.

2.2. Monte Carlo Method

Many numerical problems in engineering, economics and other applied sciences are solved by generating some statistical samples by the computer. If in such way the problem is solved we say that we solve the problem by Monte Carlo method. Monte Carlo method is used when we want:

- to observe the behaviour of the objects and processes (we generate random objects and processes);
- to estimate numerical values (we repeatedly generate samples);
- to solve complex optimization problem (we use randomized algorithms).

Assume we need to estimate a numerical value, for example, expectation $l = E(f(z))$. Assume we have simulation data

$$z^1, z^2, ..., z^N \tag{1}$$

which is independent and distributed according to the same law with the density function p. Assume that $|l| < \infty$. Then the estimate of the sample $\{z^i\}$ expectation is

$$\tilde{f} = \frac{1}{N}\sum_{i=1}^{N} f(z^i). \tag{2}$$

Let variance of $f(z)$ is finite and equal to σ^2. Then for sufficiently large N a random variable $\tilde{f}$ is distributed approximately by $N(l, \sigma^2 / N)$ law (it follows from the central limit theorem). If variance σ^2 isn't known, then it may be estimated by sample variance

$$\tilde{D}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(f(z^i) - \tilde{f})^2, \tag{3}$$

which by law of large numbers tends to σ^2 as $N \to \infty$. The estimate of confidence interval of l with significance $1-\alpha$ is

$$(\tilde{f} - \eta_{1-\alpha/2}\frac{\tilde{D}}{\sqrt{N}}, \tilde{f} + \eta_{1-\alpha/2}\frac{\tilde{D}}{\sqrt{N}}), \tag{4}$$

here η_γ is $\gamma-$quantile of the distribution $N(0,1)$. Instead of the estimate of the confidence interval we may consider the estimate of the error $\tilde{D}/\sqrt{N}$ or a relative error $\tilde{D}/\tilde{f}\sqrt{N}$. The estimation of numerical values can be done more effectively by using the information of the simulation model. We may use various variance reduction techniques, for example, importance sampling.

The idea behind importance sampling is that certain values of the input random variables in a simulation have more impact on the parameter being estimated than others. If these "important" values are emphasized by sampling more frequently, then the variance of the estimate can be reduced.

Assume we seek to estimate the value of $l = E_p H(z) = \int H(z)p(z)dz$. Here H is the real function and p (named as nominal density) is the density function of a random vector z. The index p to the expectation operator denotes that it is computed with respect to p. Assume, φ is another density function such that Hp is dominated by φ.

i.e. $\varphi(z)=0 \Rightarrow H(z)p(z)=0$. Then using density function φ we may l express as $l=\int H(z)\frac{p(z)}{\varphi(z)}\varphi(z)dz=E_{\varphi}H(z)\frac{p(z)}{\varphi(z)}$. If $z^1,...,z^N \sim \varphi$, then the estimate of l is

$$\hat{l}=\frac{1}{N}\sum_{k=1}^{N}H(z^k)\frac{p(z^k)}{\varphi(z^k)}. \tag{5}$$

The ratio of densities $W(z)=f(z)/\varphi(z)$ is the likelihood ratio. The confidence interval of l with significance $1-\alpha$ is

$$(\hat{l}-\eta_{1-\alpha/2}\frac{\tilde{D}}{\sqrt{N}},\hat{l}+\eta_{1-\alpha/2}\frac{\tilde{D}}{\sqrt{N}}). \tag{6}$$

The main difficulty using importance sampling is to choose the density φ. If density φ is poorly chosen, then it will impact the accuracy of the confidence interval of the estimate. Theoretically, optimal φ^* minimizes the variance of $\hat{l}$ and so is the solution of the problem $\min_{\varphi}\tilde{D}_{\varphi}[H(z)\frac{p(z)}{\varphi(z)}]$, here $\tilde{D}_{\varphi}$ denotes variance with respect to φ.

2.3. Methods for Stochastic Problems

In the dissertation the research is restricted to problems for which the objective function and the constraints function is of one extremum, and smoothly differentiable which are expressed by expectation of random functions satisfying Lipschitz's condition. In the case of the problems, in which absolutely continuous probability measure is present, in practice it is impossible to solve them by deterministic optimization methods. However, functions which are expressed by expectation where probability measure is absolutely continuous may be approximated by statistical modelling (Monte Carlo method). In addition, we may use stochastic methods of differentiation and estimate the gradients or the derivatives of such functions also by statistic modelling.

In the dissertation the problems of stochastic equilibrium are investigated; in these problems random players' objective functions or constraints functions satisfy Lipschitz's condition only, but the scenarios of the environment in which players make decisions are distributed according to the absolutely continuous measure. To solve such problems stochastic approximation, interior point, sample average approximation (SAA) and sequential search methods are used.

The method of stochastic approximation was perhaps the first for stochastic problems, but this method isn't very applicable in practice because this method involves the regulation (which isn't always obvious in practice) of the step of optimization and sure ways how to terminate the algorithms aren't known. The interior point methods involve complex matrix computations and aren't enough developed. SAA method successfully is applied to solve multistage stochastic linear problems, but the solving of a nonlinear stochastic problem by this method is related with the application of complex numerical algorithms. So, for stochastic equilibrium search problems these methods aren't widely applied. In literature the problems of stochastic equilibrium are analyzed more theoretically and there are many open questions in practical applications. The dissertation analyses Monte Carlo algorithms of sequential search which ensure convergence, allow finding solution with proper accuracy by using reasonable computer resources.

2.4. The Bilevel Programming - BP

The bilevel programming problems or Stackelberg games are often considered as hierarchical models where one player (leader) has priority to play in the first order and he announces his decision to other player (follower) or other players (followers). In BP problems the decision variable is separated into two vectors x and y. In the upper level the leader controls the vector $x \in \Re^m$ and in the lower level the follower controls the vector $y \in \Re^n$. If we replace the lower level problem with its KKT (Karush–Kuhn–Tucker) optimality conditions, then we have one level problem. Often this transformation of BP problems is used. Stochastic bilevel problem:

(*upper level*) $\min_{x \in X, y} E(F(x, y, \omega)),$ (7)

subject to $G(x, y, \omega) \le 0,$ here y is a solution of such problem

(*lower level*) $\min_{y} E_{\omega}(f(x, y, \omega)),$ (8)

subject to $g(x, y, \omega) \le 0,$ here $x \in \Re^m, y \in \Re^n,$

here $\omega \in \Omega$ is an elementary event in the probability space (Ω, Σ, P_x), functions are such $F : \Re^m \times \Omega \to \Re$, $f : \Re^n \times \Omega \to \Re$, $G : \Re^m \times \Re^n \times \Omega \to \Re^r$, $g : \Re^m \times \Re^n \times \Omega \to \Re^s$ and satisfy some properties of differentiability, integrability and convexity, measure P_x is

absolutely continuous and may depend on *x*, i.e. it is defined by density $p:\Re^n \times \Omega \to \Re_+$, E_ω denotes expectation with respect to a random variable ω. Of course, here may be special cases of this problem, for example, random variable ω may not be present in the constraints of upper level. Functions F, G, f, g may be of the special type, for example, linear.

2.5. Non-Cooperative Games

2.5.1. The Model of the Non-Cooperative Game

The normal form of n person non-cooperative game is triplet (N, X, u), where N is a finite set of players indexed by i, $X = X_1 \times ... \times X_n$, here X_i is a set of i player's possible actions, each vector $x = (x_1, ..., x_n) \in X$ is a profile of the actions, $u = (u_1, ..., u_n)$, here $u_i : X \to \Re$ i is player's real utility (or payoff) function.

Let (N, X, u) is a normal form game and let $\Pi(B)$ denotes all probability distributions on the set B. Then the set of mixed strategies of the player i is $S_i = \Pi(X_i)$. The set of profiles of mixed strategies is $S_1 \times ... \times S_n$. $s_i(x_i)$ denotes the probability that we will choose the action x_i in a mixed strategy s_i.

2.5.2. The Concept of Nash Equilibrium in Non-Cooperative Game

In 1951 J. Nash proposed to unify the theory of cooperative and non-cooperative games, the so called Nash program, i.e. the cooperative game to consider as some certain non-cooperative game. Assume we have the cooperative game which is defined by a characteristic function f and the set of players is N. According to Nash program, it is possible to find such non-cooperative game Γ in which some Nash equilibrium may be a solution for the cooperative game f.

For the given cooperative game we may define the reduced game, i.e. a game in which one part of players play this game and the remaining part don't play. B. Peleg (1986) established how to define the solutions for a given game via reduced game properties. V. Dumskis (2004) proposed another variant of the reduced game and analysed its properties.

We present the vector of the strategies $x = (x_1, x_2, ..., x_n)$ for easy use, as $x = (x_i, x_{-i})$, here x_{-i} is the part of the strategies vector which is controlled by other

players, except i player having strategy x_i. In the game (N, X, u) the vector $x^* \in X$ is Nash equilibrium, if $\forall i, i \in N$,

$$u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*), \qquad \forall x_i \in X_i, \ (x_i, x_{-i}^*) \in X. \tag{9}$$

2.5.3. The Stochastic Nash Equilibrium

In the practical decentralized decision-making system uncertainty often appears. Thus, it is naturally to analyse Nash equilibrium in the circumstance of the uncertainty. Assume that uncertain parameters of the system are independent stochastic variables, then the stochastic decentralized decision-making system is described as such:

$i = 1,2,...,n$ – players making decisions, y_i – part of the vector which is controlled by a decision-making player i, $u_i(y_1, y_2, ..., y_n, \xi_i)$ – the objective function of the player i, here ξ_i is random vector which describes some parameters, $g_i(y_1, y_2, ..., y_n, \xi_i)$ – constraints function for the player *i*.

In general, the players seek to optimize the expectation of their objective functions where the constraints contain the expectations of other functions. Thus, we have such model:

$$\max_{y_i} E[u_i(y_1, y_2, ..., y_n, \xi_i), \quad where \quad E[g_i(y_1, y_2, ..., y_n, \xi_i)] \leq 0 \tag{10}$$

For this case Nash equilibrium is defined by (11). Feasible solution $(y_1^*, y_2^*, ..., y_n^*)$ is Nash equilibrium if it satisfies:

$$E[u_i(y_1^*, y_2^*, ..., y_{i-1}^*, y_i, y_{i+1}^*, ..., y_n^*, \xi_i)] \leq E[u_i(y_1^*, y_2^*, ..., y_{i-1}^*, y_i^*, y_{i+1}^*, ..., y_n^*, \xi_i)], \forall i \tag{11}$$

for each feasible $(y_1^*, y_2^*, ..., y_{i-1}^*, y_i, y_{i+1}^*, ..., y_n^*)$.

3. Analysis of the Models of Financial Crises

In 2007 Watanabe K. et al. proposed the method how to predict the start of the bubble. In this work we use this last method to investigate the real estate market in Lithuania during the bubble and crash in period 2000–2009.

3.1. The Mathematical Definition of the Bubbles and Crashes

Watanabe K. et al. (2007) analysed the following model. Suppose the price at time t is $P(t)$, $P_0(T_i)$ and $w(T_i)$ are uniquely determined parameters from the past T_i data points for which root-mean-square of error $F(t)$ achieves minimum (12).

$$P(t) - P(t-1) = (w(T_i) - 1)(P(t-1) - P_0(T_i)) + F(t) \quad (12)$$

The prices behave in three manners due to the value of $w(T_i)$:

1. If $w(T_i) > 1$, then the price is exponentially increasing or decreasing and $P_0(T_i)$ is the base line of exponential divergence;
2. If $w(T_i) = 1$, then the price follows random walk;
3. If $w(T_i) < 1$, then the price converges to $P_0(T_i)$ (base line).

Equation (12) was applied to real estate market data in Lithuania from 1 May 2000 until 31 December 2009. The data in this period illustrates the bubble and the crash. We find the optimal period of observation, i.e. fix T_i. For this we will investigate an ordinary autoregressive model and will estimate parameters a_j and error term $f(t)$:

$$P(t) = \sum_{j=1}^{N} a_j P(t-j) + f(t). \quad (13)$$

The optimal interval of observation is the minimum of time scale in which we cannot observe $w(T_i) > 1$ in (12) for this time series (13) with $N = 5$. For our data the value of T_i is 12 months.

$$P_{trend}(t) = w(T_i) P_{trend}(t-1) + (1 - w(T_i)) P_0(T_i) \quad (14)$$

3.2 The Mathematical Definition of the Bubbles and Crashes in Case when Two Kinds of Agents Exist

An approach proposed in Watanabe K. et al. (2007) and certified above with real data may be useful when we have two different kinds of agents, i.e. chartists and

fundamentalists. *Fundamentalists*: There are NF fundamentalists who believe that the price will eventually converge to the fundamental value. *Chartists*: There are NS chartists who behave according to the trend of price.

Because we will explore the same data, we assume that T_i is equal to 12 and $w(T_i)$, $P_0(T_i)$ have the same meanings as above. Then we modify (12) and we have (15). N=5 as above.

$$P(t)-P(t-1)=\frac{NS}{NS+NF}\sum_{h=1}^{NS}\Big(\sum_{j=t-1}^{t-1+T_i} w(T_i)[P(j)-P(j-1)]+ \\ +\frac{NF}{NS+NF}\sum_{h=1}^{NF}[P(t-1)-P_0(T_i)]q(T_i)+F(t). \tag{15}$$

Here $q(T_i)$ and $w(T_i)$ are adjustment speeds for fundamentalists and chartists. When we introduce two kinds of agents, we estimate (15) in the same manner as (12) above. In analogous way we define the trend by (16).

$$P_{trend}(t)=w(T_i)P_{trend}(t-1)+(1-q(T_i))P_{trend}(t-1). \tag{16}$$

Here $P_0(T_i)$, $q(T_i)$, $w(T_i)$, NS and NF are uniquely determined parameters from the past T_i data points for which root-mean-square of error F(t) achieves minimum in (15).

In Fig. 1 we plotted the exponential trend curve (14), price and base line for each convergent and divergent intervals. In Fig. 2 we plotted the exponential trend curve (16), price and base line for each interval. Here we obtain different pictures.

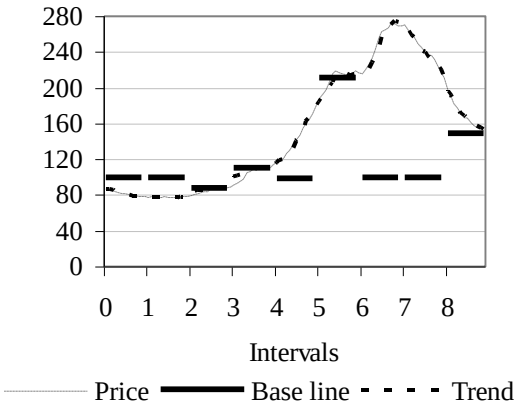


Fig. 1 Price, base line, curve of the trend

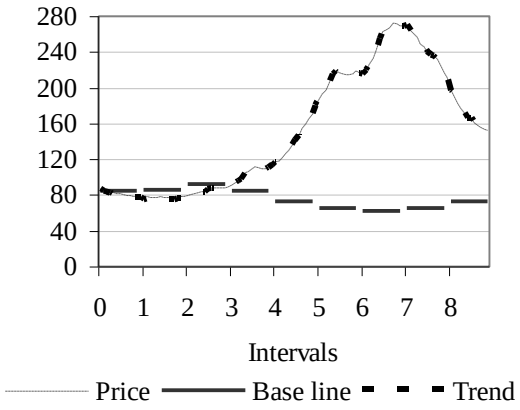


Fig. 2 Price, curve of the trend, base line (two kinds of agents)

As we see in Fig. 1, the convergence intervals are (3, 4, 6, 9) and the divergence intervals are (1, 2, 5, 7, 8). We can identify the bubble and its crash by divergence intervals. However, the interval (6), when the bubble stops growing for a while, is the

convergence interval in Fig. 1. We regard this fact as the lack of Watanabe's model; therefore we modified it by including two types of the agents and thus the interval (6) became the divergence interval (Fig. 2). The traditional heterogeneous agents model (HAM) as in Zwinkels R. C. et al. (2010) was used to analyze the behaviour of agents with the same data. In the case of the HAM model, in the start of the bubble, the part of the chartists rapidly begins growing. Thus, it validates the inclusion of the different kinds of agents into the model.

4. Stochastic Bilevel Problems

4.1 Optimization of Risk Aversion by Monte Carlo Method

Let us consider optimization of the risk aversion problem:

$$\begin{aligned} F_0(x) &= \theta \cdot E\left[f_0(x,\zeta)\right] + (1-\theta)\cdot CVaR_{\alpha_0}\left[f_0(x,\zeta)\right] \to \min_x \\ F_i(x) &= CVaR_{\alpha_i}\left[f_i(x,\zeta)\right] \le \phi_i,\ i = 1, 2, ..., m, \end{aligned} \tag{17}$$

here the conditional value at risk (CVaR) follows the definition of Rockafelar and Uryasev:

$$CVaR_\alpha(x) = \min_{v\in\Re}\left(v + \frac{1}{1-\alpha}E[f(x,\zeta)-v]^+\right) \tag{18}$$

$0 < \alpha_i < 1, i = 0,1, 2, ..., m,\ \zeta \in \Omega, (\Omega, \Sigma, P_x)$ is the probability space, $f_i : \Re^n \otimes \Omega \to \Re$, $i = 0,1, ..., m$, $[t]^+ = \max\{0, t\}$. Assume random functions $f_i : \Re^n \otimes \Omega \to \Re, i = 0,1, 2, ..., m$ satisfy Lipschitz's condition. Assume that the uncertainty is described by the absolutely continuous measure with the density function $p(x,\cdot) : \Re^m \to \Re_+$.

Thus, the problem (17) can be rewritten in a following equivalent form as the bilevel problem (see (7), (8)):

$$\begin{aligned} F_0(x) &= \theta \cdot \int_{R^l} f_0(x,z)\cdot p(x,z)dz + \\ &+ (1-\theta)\cdot\left(v_0 + \frac{1}{\alpha_0}\cdot\int_{R^l}\left(f_0(x,z)-v_0\right)^+\cdot p(x.z)dz\right) \to \min_{x,v_0} \\ F_i(x) &= \min_{v_i\in\Re}\left(v_i + \frac{1}{\alpha_i}\cdot\int_{R^l}\left(f_i(x,z)-v_i\right)^+\cdot p(x,z)dz\right) \le \phi_i, i = 1, 2, ..., m. \end{aligned} \tag{19}$$

Let us introduce the Lagrange function $F_0(x)+\sum_{i=1}^{m}\lambda_i \cdot F_i(x)$. In order to compute Lagrange function we must solve the lower level problem (18). Let us introduce the extended Lagrange function (LF)

$$\begin{aligned} &L(x,\lambda,v) \equiv El(x,\lambda,v,\zeta) = \\ &= \theta \cdot \int_{R^l} f_0(x,z)\cdot p(x,z)dz + \\ &+ \sum_{i=0}^{m}\lambda_i \cdot \left(v_i - \phi_i + \frac{1}{\alpha_i}\cdot \int_{R^n} \left(f_i(x,z)-v_i\right)^+ \cdot p(x,z)dz \right), \end{aligned} \tag{20}$$

and the extended constraints functions (CF)

$$L_i(x,v) \equiv El(x,v,\zeta) = v_i - \phi_i + \frac{1}{\alpha_i}\cdot \int_{R^n} \left(f_i(x,z)-v_i\right)^+ \cdot p(x,z)dz. \tag{21}$$

LF may be treated as expectation of the stochastic Lagrange function

$$l(x,\lambda,v,\zeta) = \theta \cdot f_0(x,\zeta) + \sum_{i=0}^{m}\lambda_i \cdot \left(v_i - \phi_i + \frac{\left(f_i(x,\zeta)-v_i\right)^+}{\alpha_i} \right) \tag{22}$$

and CF as expectation of the random functions

$$l_i(x,v,\zeta) = v_i - \phi_i + \frac{\left(f_i(x,\zeta)-v_i\right)^+}{\alpha_i} \tag{23}$$

here $\lambda = (\lambda_0, \lambda_1, ..., \lambda_m)$, $\lambda_i \geq 0, i = 1,2,...,m$, $\lambda_0 = 1-\theta$, $\phi_0 = 0$, $v = (v_0, v_1, ..., v_m)$.

We note, that functions $l(x,\lambda,v,\zeta)$ and $l_i(x,v,\zeta), i = 1,2,...,m$, satisfy Lipshitz's condition with respect to x and v, so their subgradients exist and may be treated as stochastic gradients. Because uncertainty is described by the absolutely continuous measure, the expectations in formulas above are smoothly differentiable. If density function p satisfies Lipshitz's condition, then the probabilistic objective function is twice smoothly differentiable (Bartkute and Sakalauskas, 2007). So, LF - $L(x,\lambda,v)$ and CF - $L_i(x,v)$ are smoothly differentiable; we may express their gradients by expectations. Denote the gradient of LF by $G(x,\lambda,v) = Eg(x,\lambda,v,\zeta)$ and the gradients of CF by $G_i(x,v) = Eg_i(x,v,\zeta)$, $i = 1,2,...,m$.

Assume that $x^* \in R^n$ is the solution of the problem (17). According to Karush–Kuhn–Tucker theorem and to the rules of the differentiation of CVaR (Rockafellar and Uryasev, 2002), such $\lambda_i^* \geq 0$, $v_i^* \geq 0$, $i = 0,1,...,m$ exist that

$$G(x^*,\lambda^*,v^*) \equiv \nabla F_0(x^*) + \sum_{i=0}^{m} \lambda_i^* \cdot G_i(x^*,v^*) = 0,$$
$$\lambda_i^* \cdot (L_i(x^*,v^*) - \phi_i) = 0, \qquad i = 1,2,...,m, \tag{24}$$
$$\Pr\left(f_i(x^*,\zeta) \geq v_i^*\right) = \alpha_i, \qquad i = 0,1,...,m.$$

If random functions $f_i(x^*,\zeta), i = 0,1,...,m$ are linear, then the given problem may be transformed to the large linear programming problem (LP) and then solved by *Sample Average Approximation* (SAA) method. However, the resulting LP may be very large and to solve it would require large computing resources (Kall and Mayer, 2011). Also some functions $f_i(x^*,\zeta)$, $i = 0,1,...,m$ may be nonlinear, then transformation of the given problem to LP isn't applicable. So, the simulation sequences of Monte Carlo samples may be useful (see section 2.2).

Assume that for every $x \in \Re^n$ it is possible to generate Monte Carlo samples (1). Then, according to (2), we have Monte Carlo estimates $\widetilde{F}_0(x,v)$, $\widetilde{F}_i(x,v)$ and $\Pr_i = \frac{N_i}{N}$. Here N_i are frequencies of the events $\{v_i : f_i(x,z^j) \geq v_i\}$, $i = 0,1,...,m$, in the sample (1). According to (3), we have the estimates of variances $\widetilde{D}_0^2(x,v)$, $\widetilde{D}_i^2(x,v)$.

The estimate of the extended Lagrange function is $\widetilde{L}(x,\lambda,v) = \widetilde{F}_0(x,v) + \sum_{i=1}^{m} \lambda_i \cdot \widetilde{F}_i(x,v)$.

We approximate the gradient of the objective function and the gradients of the constraints functions by stochastic gradients (Sakalauskas, 2002). Assume that stochastic gradients $G_i(x,z^j)$ of random functions $L(x,\lambda,v)$ and $L_i(x,v)$ are obtained, i.e. vectors $G_i(x,z^j)$, for which $EG_i(x,\zeta) = \nabla F_i(x)$, $i = 0,1,...,m.$ For example, stochastic gradient may be obtained as the subgradient of the random function: $G_i(x,\zeta) = \partial F_i(x,\zeta)$, $i = 0,1,...,m.$ The estimates of the gradients of the objective and constraints functions are $\widetilde{g}_0(x,v)$, $\widetilde{g}_i(x,v)$.

The estimate of the stochastic gradient of Lagrange function is:

$$q(x,\lambda,v) = \widetilde{g}_0(x,v) + \sum_{i=1}^{m} \lambda_i \cdot \widetilde{g}_i(x,v) \equiv \frac{1}{N} \sum_{j=1}^{N} Q(x,\lambda,v,z^j), \tag{25}$$

here

$$Q(x,\lambda,v,z^j)=\theta\cdot G_0(x,z^j)+(1-\theta)\cdot G_0^{\#}(x,v,z^j)+\sum_{i=1}^{m}\lambda_i\cdot G_i^{\#}(x,v,z^j).$$

Of course, $EQ(x,\lambda,v,\zeta)=Eq(x,\lambda,v)=\nabla_x L(x,\lambda,v)$.

The covariance matrix is:

$$A(x,\lambda,v)=\frac{1}{N}\sum_{j=1}^{N}\left(Q(x,\lambda,v,z^j)-q(x,\lambda,v)\right)\cdot\left(Q(x,\lambda,v,z^j)-q(x,\lambda,v)\right)^T. \tag{26}$$

4.1.1. Stochastic Algorithm of Optimization

In order to find the solution of the problem (17) or (24) Monte Carlo estimates $\tilde{F}_0(x,v)$, $\tilde{F}_i(x,v)$, $\Pr_i$, $\tilde{D}_0^2(x,v)$, $\tilde{D}_i^2(x,v)$, $\tilde{L}(x,\lambda,v)$, $\tilde{g}_0(x,v)$, $\tilde{g}_i(x,v)$ may be used. Here we'll use stochastic variable metric method (SVM), which ensures faster convergence.

Assume that the initial point $x^0\in\Re^n$ and vectors $\lambda^0, v^0\in\Re_+^m$ are given, also, that at this point sample (1) of initial size N^0 is generated and Monte Carlo estimates are computed. Let us define such SVM procedure:

$$\begin{gathered}x^{t+1}=x^t-(A^t)^{-1}\cdot q(x^t,\lambda^t,v^t),\\ \lambda_i^{t+1}=\max[0,\lambda_i^t+\gamma_i(\tilde{L}_i(x^t,v)+\mu_\beta\cdot\tilde{D}_i(x^t,v))],\ i=1,2,...,m,\\ v_i^{t+1}=v_i^t+\pi_i\cdot\left(\frac{\Pr_i}{\alpha_i}-1\right),\end{gathered} \tag{27}$$

here $\gamma_i>0, \pi_i>0$ are normalizing multipliers, μ_β is β-quantile of standard normal distribution. The covariance matrix (26) is used to change optimization metric.

Generally, the Monte Carlo estimates in (27) are random. We note that it isn't necessary to generate the sample (1) of large size at the beginning of the search for the solution and it is more important to have large size samples (1) only in the moment of the evaluation of the optimum. Therefore, the size of sample, which ensures the convergence, is chosen according to the rule (Sakalauskas, 2002):

$$N^{t+1}=\frac{\chi^2(\omega,n)}{(q^t)^T\cdot(A^t)^{-1}\cdot(q^t)}, \tag{28}$$

here ω is quantile of the distribution χ^2 with n degrees of freedom.

The termination of the algorithm may be done in a statistical way, i.e. by testing the hypothesis about the equality of the gradient of Lagrange function to zero and

validity of conditions (24). Thus, the hypothesis of the optimality may be accepted at point x with significance μ, if the inequality is valid:

$$(N-n)\cdot(\tilde{\nabla}_x L)\cdot A^{-1}\cdot(\tilde{\nabla}_x L)\le\chi^2(\mu,n) \tag{29}$$

here $\chi^2(\mu,n)$ is μ-quantile of the distribution χ^2 with n degrees of freedom, $\tilde{\nabla}_x L=\tilde{\nabla}_x L(x,\lambda)$ is the estimate of the gradient of the function LF and *A* is normalizing matrix (26) at point (x,λ), *N* is the size of sample (1). The algorithm may be stopped if the hypothesis of the optimality (29) isn't rejected and the constraints conditions (30) vanish with given probability β

$$\tilde{F}_i(x,v)+\mu_{\beta_i}\cdot\tilde{D}_i(x,v)/\sqrt{N}\le 0,\quad i=1,2,...,m, \tag{30}$$

the lengths of confidence intervals of the objective and constraints functions (see (4)) don't exceed given accuracy ε_i:

$$2\eta_{\beta_i}\cdot\tilde{D}_i(x,v)/\sqrt{N}\le\varepsilon_i,\qquad i=0,1,...,m \tag{31}$$

η_{β_i} is β_i quantile of the standard normal distribution, parameters v are well chosen to estimate CVaR:

$$\left|\mathrm{Pr}_i-\alpha_i\right|\le\eta_\beta\cdot\sqrt{\frac{\mathrm{Pr}_i\cdot(1-\mathrm{Pr}_i)}{N}},\qquad i=1,2,...,m. \tag{32}$$

If conditions (29)–(32) are valid, then optimization may be stopped with affirmation of finding the optimum of the proper accuracy. If one of these conditions isn't valid, then next sample must be generated and the optimization be continued.

4.1.2. Testing of Algorithm

The sequential simulation-based approach for stochastic programming with *CVaR* (27) has been tested by Monte-Carlo method simulating the piecewise-linear test functions (33):

$$f_i(x,\zeta)=\max_{1\le k\le k_n}\left(a_{0,k}+\sum_{j=1}^{n}a_{j,k}\cdot(x_j+\zeta_j)\right),\qquad 0\le i\le m, \tag{33}$$

where coefficients $a_{j,k}$ were chosen randomly. For each number of variables $n=2,5,10,20,50,$ the sample from *M*=100 sets of normally distributed coefficients $a_{j,k}$ has been simulated with the following values $a_{0,0}=\vartheta,\ a_{1,0}=1+3\cdot\vartheta,\ a'_{0,k}=2\vartheta,\ a'_{1,k}=\vartheta,$

$a_{j,k} = a'_{j,k} - \frac{1}{k_n} \cdot \sum_{k=1}^{k_n} a'_{j,k}$, $1 \le k \le k_n$, $m = 0{,}1$, where ϑ is standard normal. Details of the test problem are given in Table 1.

It would be noted that chose class of the functions (33) is universal because every convex function may be approximated by the functions of the type (33) by picking enough large number k_n and choosing suitable coefficients $a_{j,k}$.

The confidence levels of CVaR have been taken $\alpha_0, \alpha_1 = 0{,}1$, the value of ϕ_1 in CVaR constraint is given in Table 1. Termination conditions in (29)–(32) have been tested with probability 0.95.

Next, the Expected Value Solution (EVS) of the problem (19) has been obtained for corresponding test functions (33), where variables ζ_k were distributed normally $N(0{,}0.5)$. The EVS of CVAR threshold variables v_0 and v_1 are given in Table 1. The EVS obtained has been taken as an initial approximation in the method defined by (27).

Table 1 Data of test functions and the results of optimization

n	v_0	v_1	ϕ_1	ε	k_n	Min number of iterations	Max number of iterations	Average number of iterations
2	3.34	3.46	4.5	0.015	5	8	16	9.4
5	4.93	6.82	7	0.1	11	8	22	12.46
10	8.33	8.63	9	0.1	21	6	14	7.7
20	12.42	9.59	11.5	0.075	31	14	53	26.6
50	20.84	14.01	15	0.1	76	44	88	59.9

Some results of optimization by this method are given in Table 1. The maximal, the minimal and an average numbers of iterations needed for termination of the algorithm (27) (where the test functions (33) depending on variables' number n and admissible length of confidence interval ε) are given in this table. All test problems in samples have been solved at the proper accuracy ε, i.e. terminated according to the criteria (29)–(32) after a certain number of iterations.

The graphs of CVaR optimization results are presented in Table 2 (n=10, 20, 50). The averaged dependencies of the objective function value as well as the probabilities of CVaR on the number of iterations are given in the first three columns of first row for each case of the number (10, 20, 50) of the variables, illustrating the convergence of the approach developed. The frequency of termination, the change of the

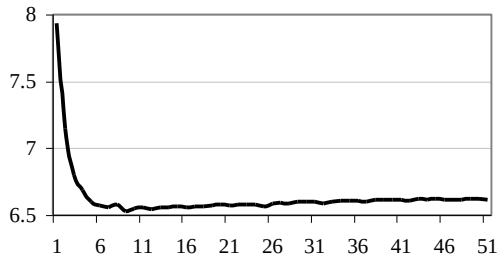
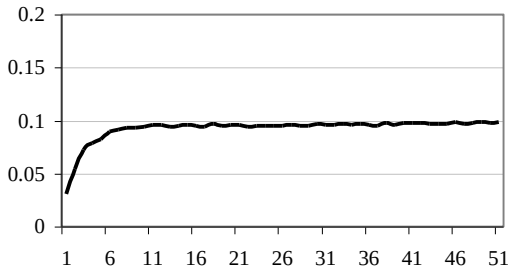
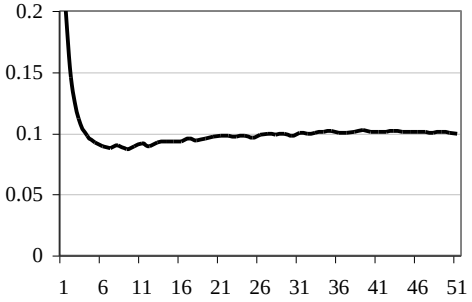
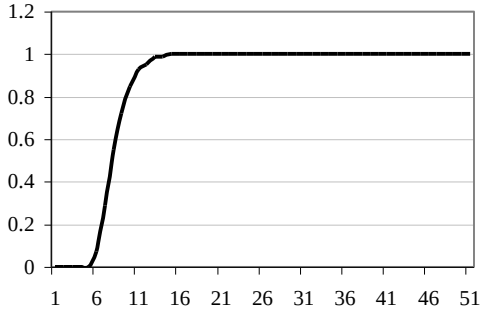
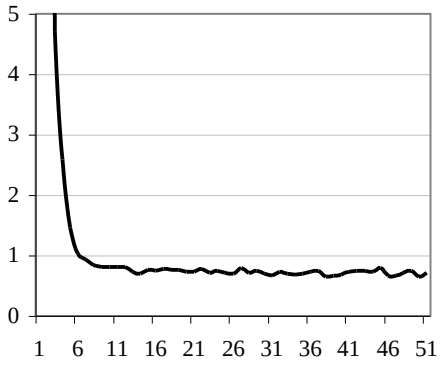
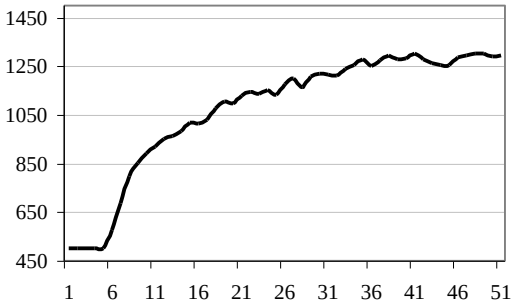
ratio of Hotelling criterion and the corresponding quantile of χ^2 distribution (which shows how this ratio tends to the critical termination value 1) depending on the number of iterations are presented in the second row for each case of the number (10, 20, 50) of the variables. The averaged number of the size of Monte Carlo sample is given for each iteration in the bottom right cell, which shows the adaptation of this sample during the optimization process. Thus, the results of Monte Carlo simulation illustrate the convergence of this approach and ability to solve the stochastic programming problems, where CVaR is included into both objective function and constraints, at proper accuracy treated in a statistical manner.

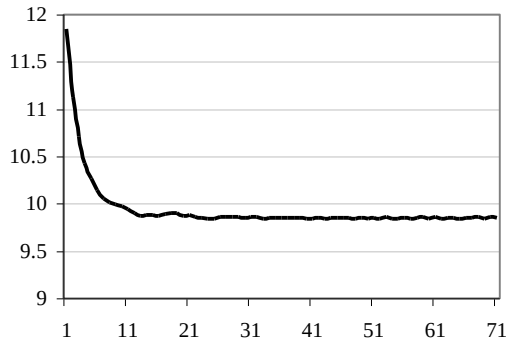
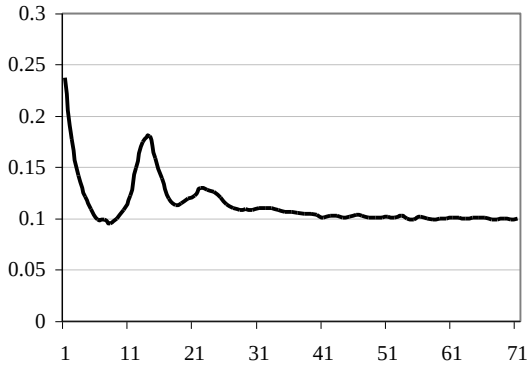
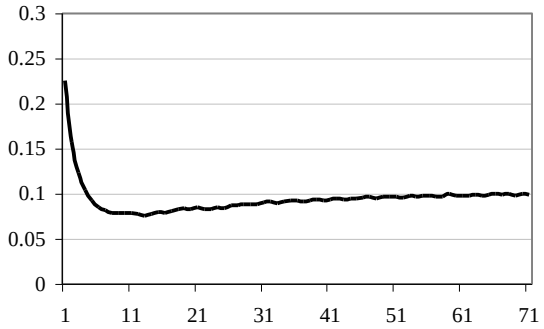
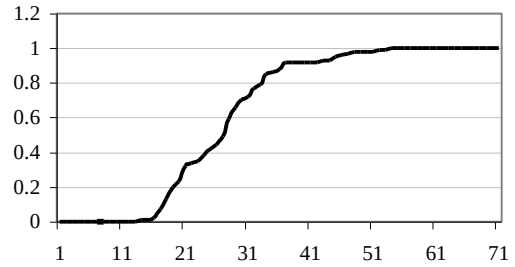
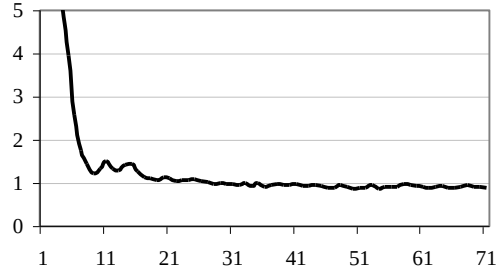
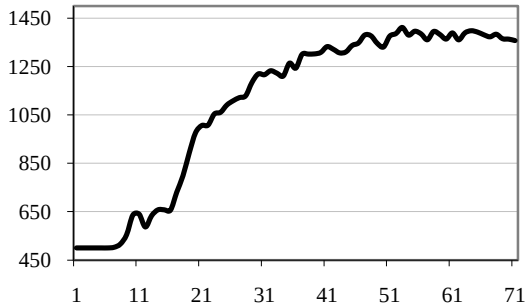
It is obvious that when solving the problem at more accuracy more iterations and greater volume of Monte Carlo sample will be needed. For example, when the number of variables was 2 and accuracy $\varepsilon = 0.015$, no less iterations and no lesser volume of Monte Carlo sample comparing with the case when the number of the variables was 5 and accuracy $\varepsilon = 0.1$ were needed.

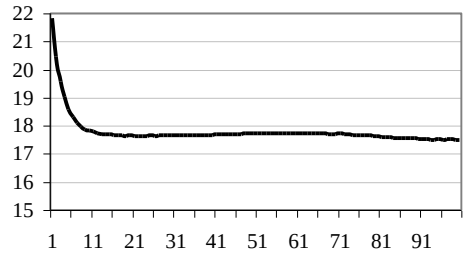
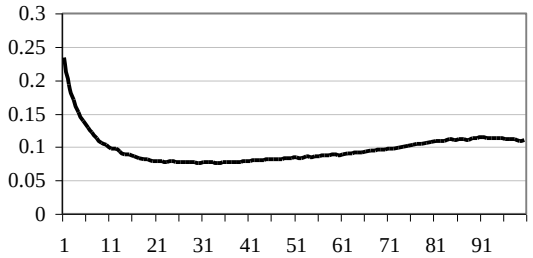
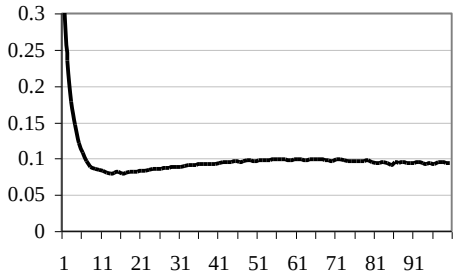
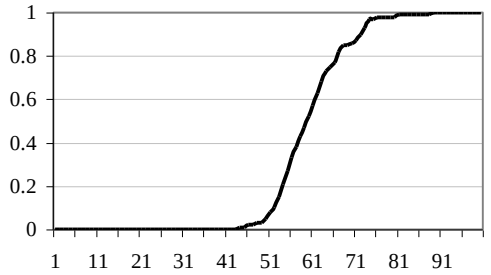
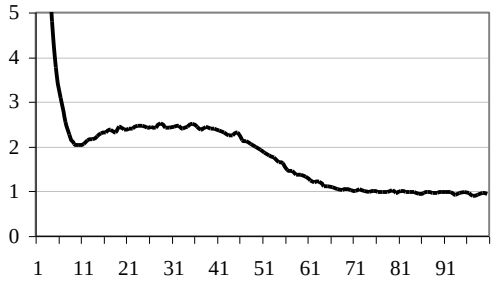
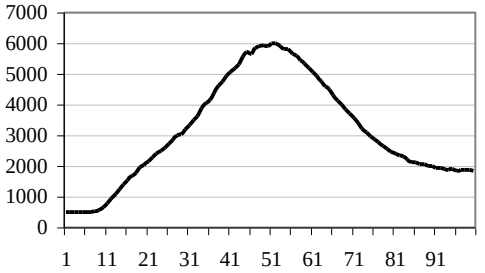
The dependence of the created algorithm on the number of the variables wasn't investigated. However, it may be noted, that when the number of the variables is increasing then the time which is needed to solve the optimisation problem is increasing too.

An investigation of depicted dependencies allows to observe the adaptation of the created algorithm. Indeed, according to (31) it is clear that the length of the confidence interval depends on the variance of the objective function. Because in the case when n=50 such objective functions, whose variances in an area of an optimum was lesser than in initial iterations, occurred, i.e. in this area they were flat; this determined the reduction of the volume of the sample in the moment of the termination (see Table 2, Monte Carlo sample size). This fact may be avoided by regulating the maximal value of the volume of Monte Carlo sample.

Table 2 The graphs of CVaR optimization results (number of the variables is 10, 20, 50)

Number of the variables (n=10)		
Objective function	Frequency $\mathbf{Pr_0}$	Frequency $\mathbf{Pr}_1$
		
Frequency of termination	Hotelling termination criterion	Monte Carlo sample size
		

Number of the variables (n=20)		
Objective function	Frequency $\mathbf{Pr_0}$	Frequency $\mathbf{Pr}_1$
		
Frequency of termination	Hotelling termination criterion	Monte Carlo sample size
		

Number of the variables (n=50)		
Objective function	Frequency $\mathbf{Pr_0}$	Frequency $\mathbf{Pr}_1$
		
Frequency of termination	Hotelling termination criterion	Monte Carlo sample size
		

4.2. Importance Sampling Method for the Two Stage Stochastic Linear Problem

As we see in section 2.2, application of the importance sampling method to evaluate the numerical value leads to the problem of minimizing of the variance. If additionally we want not only to evaluate the numerical value but to minimize that value, we have two minimizing problems, which are interdependent. Thus, we have the stochastic bilevel problem (see (7), (8)). We'll apply this approach to solve the two stage stochastic linear problem (34).

$$\begin{gathered} F(x) = c \cdot x + E\left\{\min_{r}[a \cdot r \mid W \cdot r + T \cdot x \le h, \quad r \in R_+^m]\right\} \to \min, \\ Ax = b, \quad x \in \Re_+^n, \end{gathered} \tag{34}$$

where matrices *W, T, A* and vectors *c, a, h, b* are of appropriate size, the vector *h* is the multi-normal random $N(\tau, \Sigma)$, τ is the mean vector, Σ is the covariance matrix, $p(h, \tau)$ is the density function. Denote the density function of the normally distributed vector according to $N(y, \Sigma)$ by $\varphi(\cdot, y) : \Re^m \to \Re_+$ and apply it to change the measure, changing vector y in order to decrease the variance. Denote

$$f(x,h) = \min_{\substack{W \cdot r + T \cdot x \le h \\ r \in R_+^m}} a \cdot r. \tag{35}$$

After simple manipulations we get:

$$F(x) \equiv c \cdot x + \int_{\Omega} f(x,h) \cdot q(y,h) \cdot \varphi(h,y) dh \tag{36}$$

where

$$q(y,h) = \frac{p(h,\tau)}{\varphi(h,y)} = \exp\left(-(\tau - y)^T \cdot \Sigma^{-1} \cdot (\tau + y - 2h)\right). \tag{37}$$

Correspondingly:

$$D(x,y) = \int_{\Omega} \frac{\left(f(x,h) \cdot p(h,\tau)\right)^2 \cdot}{\varphi(h,y)} dh. \tag{38}$$

It is easy to make sure that the gradient of variance (38) with respect to the parameter vector y is as follows:

$$\begin{aligned} H(x,y) &= \frac{\partial D(x,y)}{\partial y} = \int_{\Omega} (h - y) \cdot \Sigma^{-1} \cdot d(x,y,h) \cdot \varphi(h,y) dh = \\ &= E(h - y) \cdot \Sigma^{-1} \cdot d(x,y,h), \end{aligned} \tag{39}$$

where

$$d(x,y,h)=\left(f(x,h)\cdot p(h,\tau)\right)^2\cdot\frac{d\varphi(h,y)}{dy}. \tag{40}$$

The gradient of the objective function (34) with respect to x is presented as an expectation of the vector function. Indeed, according to duality of the objective function, it may be expressed as:

$$F(x)=c\cdot x+E\left\{\max_e[(h-T\cdot x)\cdot e\,|e\cdot W^T+a\geq 0,\quad e\in R_+^d]\right\}.$$

The gradient of the objective function is:

$$\frac{\partial F(x)}{\partial x}=Eg(x,h) \tag{41}$$

where $g(x,h)=c-T\cdot e^*$. Here e^* is the solution of the dual linear problem:

$$(h-T\cdot x)^T\cdot e^*=\max_e[(h-T\cdot x)^T\cdot e\,|\,e\cdot W^T+a\geq 0,\quad e\in R_+^d].$$

Assume that for every feasible solution $r\in D\subset R^n$ it is possible to generate Monte Carlo sample (1) of some size according to the density function $\varphi(h,y)$.

The estimates are computed according to (42)–(44):

$$\tilde{F}(x,y)=\frac{1}{N}\sum_{j=1}^{N}f^j \tag{42}$$

$$\tilde{G}(x,y)=\frac{1}{N}\sum_{j=1}^{N}g^j \tag{43}$$

$$\tilde{H}(x,y)=\frac{1}{N}\sum_{j=1}^{N}(z^j-y)\cdot\Sigma^{-1}\cdot d^j \tag{44}$$

where $f^j=f(x,z^j)$, $g^j=g(x,y,z^j)=g(x,z^j)\cdot q(y,z^j)$, $d^j=d(x,y,z^j)$, $1\leq j\leq N$.

Assume certain initial approximations $x^0\in\Re^n$, $y^0=\tau$ are given, certain initial sample size N^0 is chosen and random sequence $\{x^t,y^t,N^t\}_{t=0}^{\infty}$ is defined according to (45):

$$\begin{aligned}x^{t+1}&=x^t-\rho\cdot\tilde{G}_\varepsilon(x^t,y^t)\\ y^{t+1}&=y^t-\alpha\cdot\tilde{H}(x^t,y^t)\\ N^{t+1}&=\frac{1}{b^t}\cdot N^t\end{aligned} \tag{45}$$

where $\tilde{G}_\varepsilon(x^t,y^t)$ is the ε-projection of the stochastic gradient to the feasible set (see definition in Sakalauskas and Žilinskas, 2008), $\rho>0,\alpha>0,b^t>0$ are certain parameters of the method.

It can be proved by martingale approach that under an appropriate choice of parameters of the method the sequence (45) converges almost sure to the solution of the problem (34) (see Sakalauskas, 2002, 2004). Note that the choice of Monte Carlo sample size according to (45), where sample size is chosen inversely proportional to the square of stochastic gradient, allows us to solve the problem by simulating a reasonable volume of random samples.

Note that the necessary optimality condition means that the gradient of the objective function is equal to zero in the optimum. Hence, the optimization is terminated when the gradient becomes less than a certain admissible value. Indeed, the iterative process can be stopped by testing a statistical hypothesis on the equality of the gradients of the objective function to zero, and, besides, when the confidence interval of the estimate of the objective function becomes of proper length. Since the sample size increases during optimization, the distribution of the Monte Carlo estimates is approximated by the normal law; this enables us to construct statistical termination rules similar to (29) and (31) (Sakalauskas, 2002; Sakalauskas and Žilinskas, 2008).

The importance sampling approach developed has been explored by computer simulation solving the two stage stochastic linear optimisation test problem (see details in Sakalauskas and Žilinskas, 2008). The dimensions of the task are as follows: the first stage has 10 rows and 20 variables; the second stage has 20 rows and 30 variables. The estimate of the optimal value of the objective function given is 182.94234 ±0.066.

The test problem has been solved 50 times by the approach developed with the following parameters: $\rho = 0.0005, \alpha = 0.1,$ b^t was chosen by the way described in (Sakalauskas and Žilinskas, 2008). The number of iterations has been fixed when the termination conditions were satisfied for the first time (the statistical hypothesis of gradient equality to zero isn't rejected with probability 0.95 and the confidence interval of the objective function estimate doesn't exceed the admissible value $\varepsilon = 2$).

The averaged dependencies of the objective function value in regard with the number of iterations are depicted in Fig. 3, obtained by stochastic sequential optimisation with (*Import*) and without (*Classic*) the importance sampling. One can see in Fig. 3 the convergence of the approach considered in both cases. Dependency of frequency of termination is depicted in the Fig. 4, which illustrates the decrease of the

number of iterations to terminate the algorithm using the importance sampling. The total size of Monte Carlo samples applied for optimization depicted in Fig. 5 and the Monte Carlo sample size at each current iteration in Fig. 6, which shows that the importance sampling enables to decrease the amount of samples required for optimisation almost twice.

Thus, the results obtained allows to conclude that importance sampling enables to decrease the number of iterations needed to achieve the termination conditions as well as to decrease the Monte Carlo sample size at each iteration.

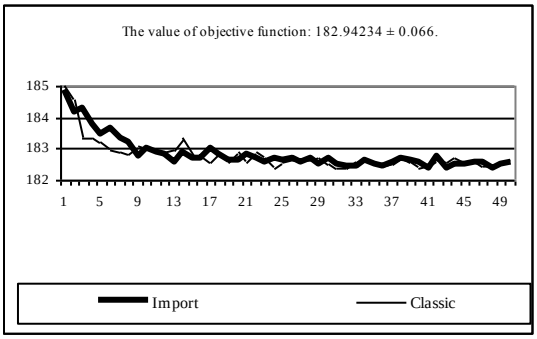


Fig. 3 Objective function value

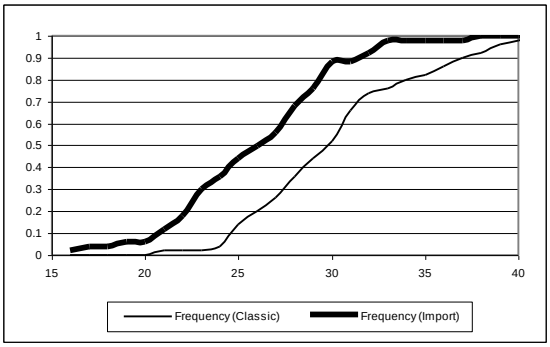


Fig. 4 Frequency of termination

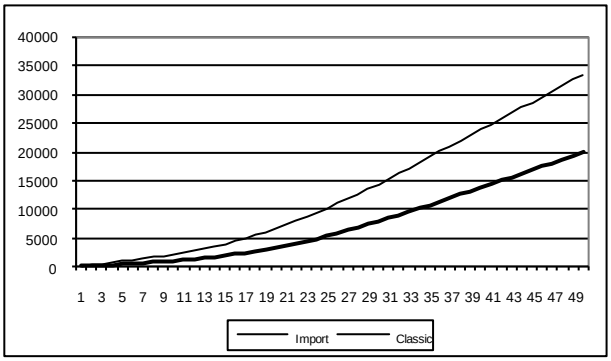


Fig. 5 Total size of Monte Carlo samples

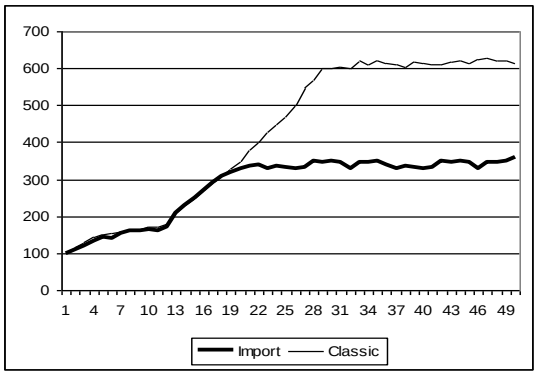


Fig. 6 Monte Carlo sample size

5. The Stochastic Nash Equilibrium Problem

5. 1. The Search for Stochastic Nash Equilibrium

We may consider the problem (46) in a general form as the non-linear Nash equilibrium problem:

$$F_j(x_1, x_2, ..., x_n) = Ef_j(x_1, x_2, ..., x_n, \xi) \to \min_{x_j \in D_j \subset R^{n_j}} \tag{46}$$

where $\xi \in \Omega$ is an elementary event in the probabilistic space (Ω, Σ, P_x), functions $f_j : \Re^n \times \Omega \to \Re$ satisfy some integrability, differentiability and convexity properties, the

measure P_x is absolutely continuous and may depend on *x*, i.e. it is described by the density function $p:\Re^n \times \Omega \to \Re_+$.

If the vectors in (46) are discrete, then functions of players $F_j(x)$ may be expressed as expectations of functions $f_j(x,\cdot)$ and the problem (46) is solved as the non-linear programming problem (often very large). To solve the problems of type (46) where random variables are continuous, generally, an assumption is drawn that it is possible to construct finite sequences of realizations of random variable ξ at every point $x \in D \subset \Re_+^n$, here $D = D_1 \otimes D_2 \otimes ... \otimes D_m \in \Re^n$, and to compute corresponding values of functions f_j and their gradients. Then it isn't difficult to estimate by Monte Carlo method the value of the objective function and the value of its gradient of the problem (46) (they are expressed by means of the expectations).

Assume that it is possible to construct Monte Carlo sample (1) of certain size N^0, where z^k are independent random variables equally distributed according to the density function $p(x,\cdot):\Omega \to \Re_+, x \in D \subset \Re^n$. Then, the estimates of the objective functions $\widetilde{F}_j(x), 1 \le j \le n$ and their variances $\widetilde{D}_j^2(x), 1 \le j \le n,$ are computed. Next, with the same Monte Carlo sample without essential computations, the estimates $\widetilde{g}_j(x), \quad 1 \le j \le n,$ of gradients of objective functions may be computed. Also, with the same Monte Carlo sample covariance matrix $A(x)$ is computed. Assume the initial point $x^0 \in D \subset \Re^n$ is given. The stochastic procedure based on gradients for search of the equilibrium is constructed as:

$$x_j^{t+1} = x_j^t - \rho_j \cdot \widetilde{g}_j(x^t) \tag{47}$$

here $\rho_j > 0, 1 \le j \le n$, are some multipliers by which the length of gradient's step is regulated. The length of the step may be tuned in the experimental way. In procedure (47) the size of Monte Carlo sample is regulated by the rule (see Sakalauskas, 2000, 2002):

$$N^{t+1} = \min\left(\max\left(\left[\frac{n \cdot Fish(\gamma, n, N^t - n)}{\rho \cdot (\widetilde{g}(x^t)^T \cdot (A(x^t))^{-1} \cdot (\widetilde{g}(x^t)} \right] + n, N_{\min} \right), N_{\max} \right) \tag{48}$$

here $Fish(\gamma, n, N^t - n)$ is Fisher distribution with $(n, N^t - n)$ degrees of freedom, γ-quantile. In order to avoid large fluctuations of size of the sample, the minimal and

maximal sizes of the sample are taken ($N_{min} \approx 20$ .. 50 and $N_{max} \approx 1000$ .. 5000). Note that the value N_{max} may be taken according to the conditions of the estimate of the confidence interval of the objective function. The rule (48) assures that the sequence obtained by (47) converges (Sakalauskas, 2000, 2002).

The test of the optimality of the solution is performed at every iteration, i.e. first, testing the hypothesis about gradient's equality to zero and, second, is the size of sample adequate to estimate the objective function with proper accuracy. Formally speaking, the testing according to Hottelig's T^2 statistics is performed, i.e. the hypothesis on optimality at point x^t may be accepted with probability $1 - \mu$, if

$$T_t^2 \equiv (N^t - n) \cdot (\tilde{g}(x^t))^T \cdot (A(x^t))^{-1} \cdot (\tilde{g}(x^t)) / n \le Fish(\mu, n, N^t - n). \qquad (49)$$

Next, according to asymptotic normality we may decide that the objective function is computed with accuracy ε, if limits of the confidence interval don't exceed the proper accuracy limit.

$$\eta_\beta \cdot \tilde{D}_j(x^t) / \sqrt{N^t} \le \varepsilon \qquad (50)$$

here η_β is β - quantile of normal distribution.

While one of conditions (49) or (50) isn't valid, the procedure (47) is repeated at point which was computed by (47), where size of the sample (1) is adjusted by (48).

5.2. Stochastic Nash Equilibrium Model of Electricity Market with Initial Contracts

Let us consider an electricity spot market with M generators, which compete one with each other in dispatching electricity. Assume, the inverse demand function is $p(Q, \zeta(\omega))$, where the market price is $p(Q, \zeta(\omega))$, Q is the total electricity supply to the market and $\zeta : \Omega \to \Re$ is a random variable which describes the uncertainty of demand (see, more details in Xu and Zhang, 2013). Each i^{th} generator chooses the quantity of electricity q_i to be produced and supplied for dispatch, $1 \le i \le M$. Then the expected profit of i^{th} generator is $R_i(q_i, Q_{i-1}) = E[q_i p(Q, \xi) - C_i(q_i) + H_i(p(Q, \xi))]$, where $Q_{i-1} = Q - q_i$, Q is total bids supplied to market by all competitors, $C_i(q_i)$ denotes the cost of producing the quantity q_i of electricity by i^{th}'s generator, $H_i(p(Q, \zeta))$ denotes the payments related to contracts, which generators sign with retailers before entering the spot market, $1 \le i \le M$. The contracts are used to hedge risks arising due to uncertainty of demand in

the market. By selling a call option at a strike price *f*, i^{th} generator pays $w_i \cdot (p - f)$ to the contract holder, if $p > f$, but no payment is made, if $p \le f$, here w_i is the quantity signed by generator on contract at strike price *f*, i.e. $H_i(p(Q,\xi)) = -w_i \max(p(Q,\xi) - f, 0)$.

The decision-making problem of i^{th} generator is to choose an optimal quantity q_i so that its expected profit would be maximized. Hence, assuming every generator to be maximizing its profit, the competition of generators in the electricity market is presented as a stochastic Nash equilibrium problem:

$$-E\left(q_i\, p(Q,\xi) - C_i(q_i) - w_i \max(\, p(Q,\xi) - f, 0)\right) \to \min_{0 \le q_i \le q_{i,\max}},$$

where $q_{i,\max}$ is the maximal generation capacity of i^{th} generator, $1 \le i \le M$. Let us consider, for example, the competition of $M = 3$ generators with the inverse demand function $p(q,\zeta) = \alpha \cdot \zeta + \alpha_0 - q$, $\alpha = 20, \alpha_0 = 30$, strike price is $f = 22$, ζ is a random variable uniformly distributed in [0,1]. Quantities of call options and cost functions for the three generators are $w_1 = 10$, $w_2 = 8$, $w_3 = 10$, $C_1(q_1) = q_1^2 + 2q_1$, $C_2(q_2) = 2q_2^2 + 2q_2$, $C_3(q_3) = 2q_3^2 + 3q_3$. Here $F_\mu = Fish(\mu, n, N^t - n)$.

Table 3 Computed and exact equilibriums

t	q_1	q_2	q_3	Q	$E[p(Q,\xi)]$	R_1	R_2	R_3	N^t	T_t^2 / F_μ
1	10	10	10	30	10.274	-17.26	-117.26	-127.26	2	$6.4*10^3$
2	7.827	5.827	5.727	19.382	20.708	65.91	25.69	35.81	9	70.263
3	7.784	5.132	4.634	17.550	22.324	70.64	30.05	46.60	204	2.474
4	7.983	5.000	4.250	17.233	22.547	72.65	30.62	46.95	2283	1.004
5	8.151	4.961	4.079	17.191	22.959	74.33	30.70	48.14	926	0.999
6	8.358	5.022	4.036	17.416	22.505	73.38	30.02	46.14	4085	1.002
7	8.440	4.992	3.968	17.401	22.793	74.58	30.21	47.05	10316	0.970
8	8.532	5.011	3.963	17.506	22.540	74.79	30.58	46.03	3656	1.000
9	8.552	4.980	3.936	17.468	22.384	74.37	30.46	45.31	2538	1.000
10	8.544	4.944	3.906	17.394	22.609	74.93	30.48	46.08	39748	0.252
Analytic solution										
q^*	8.642	4.980	3.923	17.545	21.687	75.70	30.70	46.50		

The example considered has been solved starting with the set of strategies $q = (10,\ 10,\ 10)$, afterwards simulating the Monte Carlo samples choosing the strategies (q_1, q_2, q_3) in a sequential way according to (47) and adopting the sample size N^t according to (48). Results of computer simulation are presented in Table 3, where the analytically obtained solution of the problem is given for comparison. The presented results enable to conclude that only few samples have to be simulated in order to achieve the set of equilibrium strategies.

General Conclusions

The problems of search of the stochastic equilibrium which often are characterized by a hierarchic structure are encountered in various scientific fields: engineering, economics, finances and logistics. Many of applied problems of search of stochastic equilibrium are considered as problems of equilibrium search where payoff functions are convex and satisfy Lipschitz's conditions, and the random scenarios of the environment are distributed according to the discrete or continuous probability law.

The research explored unsustainable states of market, i.e. economic bubbles and their crashes. In the dissertation the mathematical model of the bubble is proposed, dealing with agents of two different types: fundamentalists and chartists; and this model was employed for exploration of Lithuanian real estate bubble.

Because often in the real situations the risk is considered, it is necessary to incorporate the criteria of the risk in the optimization problems. This can be made by involving criteria of the risk in the objective function and in the constraints, thus the resulting problem is the bilevel stochastic problem. The method of sequential Monte Carlo search (algorithm 4.1) has been created to solve this problem.

Applying the method of importance sampling for solving the two stage stochastic linear problem transforms the given problem to the stochastic bilevel programming problem. In this work the algorithm 4.2 has been created to solve this problem.

The created algorithm 5.1 for the gradient search of stochastic Nash equilibrium has been employed to solve the test problem of the electricity market with the initial contracts.

The research carried out in the dissertation enables to draw such conclusions:

1. Having included heterogeneous agents into the mathematical model of the economic bubble improves adequacy of it, i.e. if the growth of the bubble stops a little, the effect of the price divergence from base line remains.
2. The solving of the test problem where the objective function and constraints contain the conditional value at risk by the way of the algorithm 4.1 enables to find the solution of the given accuracy, i.e. checking the condition of termination of the algorithm with probability 0.95, calculating CVaR with significance level 0.1, where

a number of variables of the test functions varies from 2 to 50, an average number of iterations varies from ≈ 8 to ≈ 60, an average size of Monte Carlo samples varies from ≈ 1200 to ≈ 6000.

3. The solving of the two stage stochastic linear problems by the way of the algorithm 4.2 enables *to* reduce *approx. 15-20%* of the number of the iterations which is needed to find the solution of the given accuracy, also, the average size of the Monte Carlo sample *is almost twice less* when the algorithm has been stopped comparing with the classical algorithm.
4. The algorithm 5.1 for gradient search of the stochastic Nash equilibrium is effective because the solving of the test problem by it enabled to find the solution of the given accuracy via several (≈ 10) iterations when the size of Monte Carlo sample didn't exceed 10,000.
5. The created algorithms were investigated by the way of statistical modelling solving practical test problems, and gained conclusions in such a way may be applied in creating and investigating the algorithms for the search of the stochastic equilibrium of other problems.

List of Literature, Referenced in this Summary

1. Cheng L., Wan Z., Wang G. (2009). Bilevel Newsvendor Models Considering Retailer with CVaR Objective, *Computers and Industrial Engineering*, 57(1), p. 310–318.
2. Bartkute V., Sakalauskas L. (2007). Simultaneous Perturbation Stochastic Approximation for Nonsmooth Functions, *European Journal of Operational Research, Elsevier Science*, Vol. 181, No. 3, p. 1174–1188.
3. Dumskis V. (2004). Reduced game and its solutions, *Lietuvos matematikos rinkinys / Lithuanian Mathematical Journal*, Vol. 44, p. 609–611.
4. Kall P., Mayer J. (2011). Stochastic Linear Programming: Models, Theory, and Computation, *Springer, International Series in Operations Research and Management Science.*
5. Patriksson M., Wynter L. (1997). Stochastic Nonlinear Bilevel Programming, *Technical Report, PRISM, Universit´e de Versailles – Saint Quentin en Yvelines, Versailles, France.*

6. Peleg B. (1986). On the Reduced Game Property and its Converse, *International Journal of Game Theory*, 15, 187–200.
7. Rockafellar R.T., Uryasev S. (2002). Conditional Value-at-Risk for General Loss Distributions. *Journal of Banking and Finance*, 26/7, p. 1443–1471.
8. Sakalauskas L. (2002). Nonlinear Stochastic Programming by Monte-Carlo estimators, *European Journal on Operational Research*, Vol. 137, p. 558–573.
9. Sakalauskas L. (2004), Nonlinear stochastic optimization by Monte-Carlo estimators, *Informatica*, Vol. 15 (2), p. 271–282..
10. Sakalauskas L., Žilinskas K. (2008). Epsilon-projection method for two-stage SLP, *Lietuvos matematikos rinkinys / Lithuanian Mathematical Journal*, Vol. 48/49, special No. p. 320–326.
11. Watanabe K., Takayasu H., Takayasu M. (2007). A mathematical definition of the financial bubbles and crashes, *Physica A: Statistical Mechanics and its Applications*, Vol. 383, Issue 1, p. 120–124.
12. Xu H., Zhang D. (2013) Stochastic Nash equilibrium problems: sample average approximation and applications, *Computational Optimization and Applications*, [access via the Internet]: <http://link.springer.com/article/10.1007/s10589-013-9538-7>.
13. Zwinkels R. C. J., Saskia ter Ellen (2010). Oil price dynamics: A behavioral finance approach with heterogeneous agents, *Energy Economics*, 32, p. 1427–1434.

List of Publications on Topic of Dissertation

1. V. Dumskis. (2004) Reduced game and its solutions, *Lietuvos matematikos rinkinys / Lithuanian Mathematical Journal*, Vol. 44, p. 609–611, ISSN 0132-2818.
2. V. Dumskis, L. Sakalauskas (2009). Ponzi modelling of the real estate market, *Knowledge-based technologies and OR methodologies for strategic decisions of sustainabled development : 5th international Vilnius conference, EURO-mini conference, September 30–October 3, 2009 : Selected papers, edited by M. Grasserbauer, L. Sakalauskas, E.K. Zavadskas, Vilnius*, p. 538–543, ISBN 978-9955-28-482-6.

3. V. Dumskis, L. Sakalauskas. (2012). The mathematical definition of the bubbles and crashes, *Jaunųjų mokslininkų darbai / Journal of Young Scientists*, Vol. 4 (37), p. 212–217, ISSN 1648-8776.
4. V. Dumskis, V. Guigues, L. Sakalauskas. (2012). On Risk Aversion Optimization by Monte – Carlo Method, *International Workshop "Stochastic Programming for Implementation and Advanced Applications" (STOPROG -2012), Proceedings, July 3-6, 2012,Neringa*, p. 19–23, ISBN 978-609-95241-4-6.
5. V. Dumskis, L. Sakalauskas. (2013). Reikšmingų imčių metodas dviejų etapų stochastiniam tiesiniam uždaviniui [Importance Sampling Method for Two-Stage Linear Stochastic Problem], *Jaunųjų mokslininkų darbai / Journal of Young Scientists*, Vol. 2 (40), p. 120–124, ISSN 1648-8776. (*in Lithuanian*)
6. V. Dumskis, L. Sakalauskas. (2014). Stochastinės Nešo pusiausvyros paieškos algoritmas [The Algorithm for Search of Stochastic Nash Equilibrium], *Jaunųjų mokslininkų darbai / Journal of Young Scientists, accepted for print in 2013*, ISSN 1648-8776. (*in Lithuanian*)

Briefly about the Author

Valerijonas Dumskis graduated from Meškuičiai Secondary School of Šiauliai District in 1971. In 1971–1976 was studying mathematics at Vilnius University and was awarded the qualification of the mathematician. Since 1976 was working at the Department of Mathematics of Šiauliai University and since 1997 is working as a lecturer at the Department of Informatics of Šiauliai University. In 1978–1980 was serving in the Soviet army. In 2009–2013 was a doctoral student at the Institute of Mathematics and Informatics of Vilnius University.

E-mail: valius.du@svajone.su.lt

REZIUMĖ

Tyrimų sritis

Pusiausvyros būsena ir su ja susiję klausimai aktualūs daugeliui ekonomikos, verslo bei finansų valdymo sričių. Paprastai pusiausvyra yra kurio nors dinaminio proceso išdava. Svarbu nustatyti, 1) kurioms sąlygoms esant egzistuoja pusiausvyra ir jei ji egzistuoja, tai 2) ar ji yra vienintelė, kaip pasiekiama pusiausvyra, 3) kiek yra stabili toji pusiausvyra. Pusiausvyros paieška labai svarbi ekonominės analizės taikymams: prognozuojant baigmę ir įvertinant tinkamas modelio parametrų reikšmes, lyginant eksperimento rezultatus su modelio prognozėmis, testuojant suprojektuotą mechanizmą. Taikymuose dažnai analizuojama Nešo pusiausvyra, kurioje nė vienas iš lenktyniaujančių individų (lošėjų), besielgiančių nekooperuotai, neturi intencijos (jam nenaudinga) vienpusiškai keisti savo strategiją, kai kiti individai laikosi pusiausvyros strategijos. Jei lošėjai priima sprendimus hierarchiškai, tai turime *Stakelbergo pusiausvyrą*. Daugelyje ekonomikos ir finansų sričių susiduriama su vienokios ar kitokios rūšies neapibrėžtumu, pavyzdžiui, paklausa priklauso nuo tam tikrų atsitiktinių dydžių arba kai kurių dydžių negalima tiksliai nustatyti ar išmatuoti. Tada būtina nagrinėti stochastinę pusiausvyrą.

Praktiniuose uždaviniuose, kai esama daug skirtingų individų, tenka tirti jų elgsenos dinamiką, nustatyti pastarosios ribinius atvejus bei rasti parametrų reikšmes, prie kurių ši dinamika veda į pusiausvyrą.

Problemos aktualumas

Jei, priimant sprendimą, dalyvauja keletas lošėjų, kurių interesai nesutampa ir jie negali kooperuotis, toks lošimas vadinamas nekooperatiniu lošimu. Jame lošėjas, rinkdamasis iš jam galimų strategijų, siekia maksimizuoti (minimizuoti) savo tikslo funkciją, kurios reikšmė priklauso ir nuo kitų lošėjų strategijų. 1950 metais J. Nešas pasiūlė nekooperatiniams lošimams pusiausvyros koncepciją. Esant pusiausvyrai, pasak Nešo, nė vienas lošėjas neturi ketinimo keisti savo strategijos (jis tokiu atveju gautų tą patį arba mažiau), jei visi kiti lošėjai laikosi pusiausvyrinių strategijų.

Daugelis pusiausvyros situacijų modeliuojama dviejų lygių sprendimo priėmimo modeliais, kuriuose svarbi sprendimo priėmimo tvarka. Tie modeliai, vadinami *Stakelbergo lošimais*, kuriuose leistina aibė yra nusakoma kito, parametrinio

optimizavimo uždavinio optimaliais sprendiniais. Šio tipo modeliai taikomi įvairiose srityse, kur viršutiniame lygyje lyderis maksimizuoja (minimizuoja) savo tikslo funkciją, priklausančią tiek nuo jo, tiek ir nuo pasekėjo strategijos, be to, pasirenka strategiją, į kurią reaguojantis pasekėjas priima sprendimą apatiniame lygyje taip, kad maksimizuotų (minimizuotų) savo tikslo funkciją.

Labai dažnai reikia įvertinti ne tik išlošį ar praradimą, kurie priklauso nuo atsitiktinių dydžių, bet įvertinti ir riziką. Rizikos įvertinimo klausimai yra labai svarbūs finansų rinkose. Tada į modelį reikia įtraukti kurį nors rizikos matą: tai gali būti *rizikos reikšmė* – VaR (angl. *Value at Risk*) ar *sąlyginė rizikos reikšmė* – CVaR (angl. *Conditional Value at Risk*).

Tyrimų objektas

Disertacijos objektas – heterogeninių agentų modelio tyrimas ir taikymas stochastinės Nešo ir Stakelbergo pusiausvyrų modeliavimui Monte Karlo metodu.

Tyrimų tikslas ir uždaviniai

Darbo tikslas – nustatyti heterogeninių agentų įtaką ekonominio burbulo susiformavimui, sudaryti ir ištirti dviejų lygių stochastinio programavimo specialių uždavinių bei stochastinės Nešo pusiausvyros paieškos Monte Karlo algoritmus. Šiam tikslui įgyvendinti numatyti tokie uždaviniai:

1. Sukonstruoti ekonominio burbulo ir jo griūties matematinį modelį bei pritaikyti jį Lietuvos nekilnojamojo turto burbului ištirti.
2. Sudaryti Monte Karlo algoritmą sąlyginei rizikai optimizuoti, esant ribojimuose sąlyginei rizikai.
3. Atlikti reikšmingų imčių metodo tyrimą ir jį pritaikyti tiesinio dviejų etapų stochastinio programavimo uždaviniui spręsti.
4. Atlikti stochastinės Nešo pusiausvyros tyrimą ir sudaryti algoritmą tos pusiausvyros paieškai, įvertinti šio algoritmo elgseną.

Mokslinis naujumas

Nauji atlikto tyrimo rezultatai:

1. Pasiūlytas ekonominio burbulo ir jo griūties matematinis modelis, į kurį įtraukti dviejų skirtingų tipų agentai.
2. Sudarytas ir ištirtas stochastinio programavimo uždavinio, kurio tikslo funkcijoje ir ribojimuose yra sąlyginė rizikos reikšmė, sprendimo Monte Karlo algoritmas.

3. Sudarytas ir ištirtas reikšmingų imčių algoritmas, pritaikytas dviejų etapų tiesinio stochastinio programavimo uždaviniui spręsti.
4. Sudarytas ir ištirtas stochastinės Nešo pusiausvyros paieškos Monte Karlo algoritmas, jo stabdymui pritaikant statistinius kriterijus.

Praktinė darbo reikšmė

Praktiniai darbo rezultatai:

1. Pasiūlytas netvarios būsenos (burbulo ir jo griūties) matematinis modelis, kuris pritaikytas Lietuvos nekilnojamojo turto burbulo tyrimui.
2. Sudarytas Monte Karlo algoritmas sąlyginei rizikos reikšmei optimizuoti, kai esama ribojimų sąlyginei rizikos reikšmei; algoritmas ištirtas, sprendžiant sugeneruotus testinius uždavinius.
3. Sudarytas dviejų etapų tiesinio stochastinio programavimo reikšmingų imčių algoritmas, kuris ištirtas sprendžiant testinį uždavinį.
4. Sudarytas algoritmas stochastinės Nešo pusiausvyros paieškai; jis pritaikytas elektros tiekimo su išankstiniais sandoriais uždaviniui spręsti.

Ginamieji teiginiai

1. Į ekonominio burbulo matematinį modelį įtraukus dviejų skirtingų tipų agentus (čartistus ir fundamentalistus), gaunamas tikslesnis burbulo pradžios ir jo griūties nustatymas.
2. Stochastinės Nešo arba Stakelbergo pusiausvyros nuoseklios Monte Karlo paieškos algoritmai pasižymi geru konvergavimu ir leidžia nustatyti pusiausvyros strategiją norimu tikslumu, stabdant algoritmą pagal statistinį kriterijų.

Darbo rezultatų aprobavimas

Tyrimų rezultatai buvo pristatyti tarptautinėse mokslinėse konferencijose (3 pranešimai) ir Lietuvos mokslinėse konferencijose (4 pranešimai). Darbo rezultatai buvo paskelbti mokslo leidiniuose, referuojamuose pripažintose tarptautinėse duomenų bazėse (Lietuvos mokslo tarybos patvirtintas sąrašas): *CEEOL* ir *Index Copernicus* – 2 straipsniai ir 1 straipsnis priimtas spausdinimui, *MatSciNet* (*MathematicalReviews*) – 1 straipsnis. Tarptautinių konferencijų darbuose, įtrauktuose į Mokslinės informacijos instituto sąrašą – 1 publikacija. Recenzuojamoje Lietuvos tarptautinės konferencijos medžiagoje – 1 publikacija.

Disertacijos struktūra

Disertaciją sudaro penki skyriai, literatūros sąrašas ir priedai. **1-asis** skyrius – įvadas. **2-ame** skyriuje formuluojami pusiausvyros paieškos uždaviniai: heterogeninių agentų modelis, dviejų lygių stochastinio programavimo uždavinys, Nešo pusiausvyros modelis ir atliekamas analitinis jų tyrimas. **3-iame** skyriuje tiriami rinkos finansiniai burbulai ir jų griūtys: naudojant Ponzi schemą ir pasiūlytą finansinių burbulų ir jų griūčių matematinį modelį bei heterogeninių agentų modelį. **4-ame** skyriuje tiriami dviejų lygių stochastinio programavimo uždaviniai (uždavinys, kurio tikslo funkcijoje ir ribojimuose yra CVaR, dviejų etapų stochastinio tiesinio programavimo uždavinio sprendimas reikšmingų imčių metodu). **5-ame** skyriuje tiriamas stochastinės Nešo pusiausvyros paieškos algoritmas bei jo taikymas. Darbo pabaigoje yra pateiktos išvados, literatūra ir priedas.

Bendrosios išvados

Stochastinės pusiausvyros paieškos uždaviniai, dažnai pasižymintys hierarchine struktūra, būdingi įvairioms mokslo sritims – inžinerijai, ekonomikai, finansams, logistikai. Daugelį taikomųjų stochastinės pusiausvyros paieškos uždavinių galima formuluoti kaip pusiausvyros paieškos uždavinius su iškilomis, tenkinančioms Lipšico sąlygas išlošio funkcijomis ir atsitiktiniais aplinkos scenarijais, pasiskirsčiusiais pagal diskretųjį ar tolydųjį tikimybinius dėsnius.

Darbe ištirtos rinkos netvarios būsenos, t. y. ekonominiai burbulai ir jų griūtys. Disertacijoje pasiūlytas matematinis burbulo modelis, kuriame yra dviejų tipų agentai – fundamentalistai ir čartistai. Šiuo modeliu ištirtas Lietuvos nekilnojamojo turto burbulas.

Kadangi realiose situacijose dažnai tenka atsižvelgti į riziką, tai į optimizavimo uždavinius būtina įtraukti rizikos kriterijus. Tai galima pasiekti, įvedus rizikos kriterijus į tikslo funkciją bei ribojimus, o tokiu būdu gaunamas uždavinys yra dviejų lygių stochastinis programavimo uždavinys. Sukurtas nuoseklios Monte Karlo paieškos metodas (4.1 algoritmas) šiam uždaviniui spręsti.

Reikšmingų imčių metodo taikymas tiesiniam dviejų etapų stochastiniam programavimo uždaviniui spręsti transformuoja duotą uždavinį į dviejų lygių stochastinį programavimo uždavinį. Šiame darbe sukurtas 4.2 algoritmas šiam uždaviniui spręsti.

Sukurtas stochastinės Nešo pusiausvyros gradientinės paieškos 5.1 algoritmas buvo pritaikytas testiniam elektros rinkos su išankstiniais sandoriais uždaviniui spręsti.

Šioje disertacijoje atlikti tyrimai leidžia padaryti tokias išvadas:

1. Heterogeninių agentų įtraukimas pagerina ekonominio burbulo matematinio modelio adekvatumą, t. y. jei burbulas trumpam nustoja augti, tai kainos divergavimo nuo bazinės linijos efektas modelyje išlieka.
2. Sprendžiant 4.1 algoritmu testinį rizikos optimizavimo uždavinį, į kurio tikslo funkciją ir ribojimus įeina sąlyginė rizika, leidžia surasti duoto tikslumo sprendinį, t y., tikrinant algoritmo stabdymo sąlygą su tikimybe 0,95, CVaR skaičiuojant reikšmingumo lygmeniu 0,1, kai testinių funkcijų kintamųjų skaičius yra 2, 5, 10, 20, 50, vidutinis iteracijų skaičius kinta nuo ≈ 8 iki ≈ 60, o Monte Karlo imties dydis kinta nuo ≈ 950 iki ≈ 6000.
3. Sprendžiant 4.2 algoritmu tiesinį dviejų etapų stochastinio programavimo uždavinį, reikia *15-20%* mažiau iteracijų duoto tikslumo sprendiniui rasti, be to, vidutinis Monte Karlo imties dydis stabdymo momentu yra *beveik dvigubai mažesnis*.
4. Stochastinės Nešo pusiausvyros gradientinės paieškos 5.1 algoritmas yra efektyvus, nes juo sprendžiant testinį uždavinį, pakako kelių (≈ 10) iteracijų duoto tikslumo sprendiniui rasti, kai Monte Karlo imties dydis neviršijo 10 000.
5. Sukurti algoritmai buvo ištirti statistinio modeliavimo būdu, sprendžiant testinius ir praktinius stochastinės paieškos uždavinius, o gautos tokiu būdu išvados taikytinos sukuriant ir tiriant kitų stochastinės pusiausvyros paieškos uždavinių algoritmus.

Trumpai apie autorių

Valerijonas Dumskis 1971 metais baigė Šiaulių rajono Meškuičių vidurinę mokyklą. 1971–1976 metais studijavo Vilniaus universitete, įgijo matematiko kvalifikaciją. Nuo 1976 metų dirbo Šiaulių universitete Matematikos katedros dėstytoju, o nuo 1997 metų dirba Šiaulių universiteto Informatikos katedros lektoriumi. 1978–1980 metais tarnavo Sovietų Sąjungos armijoje. 2009–2013 metais Vilniaus universiteto Matematikos ir informatikos instituto doktorantas.

El. paštas: valius.du@svajone.su.lt