VILNIUS UNIVERSITY

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Effective Method to Obtain Terminating Proof-Search in Transitive Multimodal Logics

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VILNIAUS UNIVERSITETAS

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Efektyvus metodas baigtinei išvedimo paieškai tranzityviose multimodalinėse logikose gauti

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Abstract

The knowledge of agents is usually modelled using logic *S5*. However in some cases it is preferable to use other modal logics, for example *S4* or even its multimodal variant $S4_n$. It also can be noted, that *S5* can be trivially embedded into *S4* ([14]) and the satisfiability problem for $S4_n$ is PSPACE-complete ([35]). Although multimodal epistemic logics are capable of modelling knowledge of many different agents, they do not include interaction between them. In this dissertation one particular form of interaction is chosen: one of the agents is called the central agent, because it knows everything that is known to other agents. This interaction is essentially the same as distributed knowledge.

The main aim of this thesis is to present a sequent calculus for multimodal logic $S4_n$ with central agent axiom in which every derivation search terminates. To achieve this task, basic sequent calculus is derived from the respective Hilbert-type calculus and the cut-elimination theorem is proved. Next the obtained calculus is modified to ensure the termination of derivation search. This is done using different kind of labels: positive and negative indexes of the modality, stars of the negatively indexed modality, marks of the positively indexed modality and formula numbers. These labels are used to restrict the applications of the rules, which causes loops in derivation search trees.

Moreover, the research allowed to extend the results to other logics, therefore terminating calculi for multimodal epistemic logics K_n, T_n and $K4_n$ with central agent axiom are also presented. Although termination of proof search in the sequent calculi for K_n and T_n with central agent axiom is obtained with only little or no effort, the transitivity axiom $\Box_l F \supset \Box_l \Box_l F$ of logics $K4_n$ and $S4_n$ with central agent axiom causes much more difficulties. To solve these problems, the new terminating calculi for monomodal logics *K4* and *S4* are derived and also presented in this thesis.

Needless to say, that this thesis also proves the soundness and complete-

ness of every newly introduced calculus. It also shows, that every derivation search in each of the terminating calculi is finite.

Acknowledgements

Dedication:

> I dedicate this work to my father Juozapas Andrikonis, who once wished to complete doctoral studies, however chose to sacrifice his own aspirations in order to devote more time for raising me and my siblings.

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Contents

Introduction

Research Area and Problem relevance

Different modal logics are widely applied in computer science and artificial intelligence. One such field of application is epistemology - a science about knowledge and belief. Although the most popular epistemic logic for knowledge modelling is *S5* ([41, 54]), in some cases other logics are chosen instead, for example *S4* ([7, 8, 43]). Multimodal $S4_n$ is also chosen in favour of $S5_n$ in some other applications, e.g. in [9]. There are many cut-free systems for *S5* (they are summarized in [50]), but all of them introduce changes into the original Gentzen-type calculus. For example, some of them enrich formulas with indices (see [42]), some of them use different expressions instead of sequents (see [31, 47]). This is not the problem for logic *S4*. Finally, modal logic *S5* can be trivially embedded into *S4* ([14]) and the satisfiability problem for $S4_n$ is PSPACE-complete ([35])[1]. In this thesis logic *S4* is analysed and the results are extended to other epistemic logics *K*, *K4* and *T*.

However monomodal logic *S4* is not enough to reason about the knowledge of many agents, therefore multimodal logic must be used, but it can be only a base for discussion about multi-agent systems, because it does not include interactions between the knowledge of different agents. To deal with this peculiarity, $S4_n$ can be enriched by various interaction axioms. For example in [39], possible relations between agents in two agent systems are analysed. In [38], one particular class of interpreted systems is analysed and an interaction axiom is proposed for this system. Several interaction axioms are presented in [21]. Moreover, two possible scenarios for multi-agent systems are displayed in [38, 39, 40]:

1. A system with a central processing unit. There is one agent (called the central agent and denoted c), that knows everything what is known to

[1] The complexity for other monomodal as well as multimodal logics are summarised in [26].

other agents. To model this scenario, the axiom $\Box_a F \supset \Box_c F$[2] is added to modal logic. In this dissertation it is called the central agent axiom.

2. A system with agents of different capabilities. The agents are ordered according to the computational power and any agent with more computational power knows everything what is known to the agents with less computational power. Similar axiom $\Box_i F \supset \Box_j F$[3], where i is agent with less computational power than j, can be added to model this situation.

In this thesis the first case is analysed, because of two reasons. First of all, the results presented in this dissertation can be extended to cover the second case of interaction. And secondly, as it is shown later in the thesis, the central agent modality models the behaviour of distributed logic operator. Distributed knowledge was introduced in [24] where it was called "implicit knowledge". The name "distributed knowledge" was first used in [25] and the concept is widely analysed (e.g., [11, 12, 23, 26, 41]).

Research Objectives

A variety of methods to derive theorems in different modal logics exist and to ease the derivation search computer programs are developed. However in order for a method to be suitable for automation, it must be algorithmic. That is, it must have two basic properties. Firstly, in every step a method must provide a single action and secondly, a method must stop in both situations: if the sequent is derivable and if it is not. The main objective of this research is to develop such method for considered multimodal logic $S4_n$ with central agent axiom.

Aim of the Work and Work Tasks

Hilbert-type calculus is a usual way to define a deduction system for modal logic. In such system the needed properties of the logic are formulated as axioms and several derivation rules. Although such definition is very convenient for semantic discussion, however the derivation search process using this technique causes a lot of problems. Therefore, in this thesis Gentzen-type calculus (also known as sequent calculus) is used and the first

[2] If agent a knows F, then the central agent also knows F.
[3] If agent i knows F, then agent j also knows F.

task is to develop sound and complete sequent calculi for considered logics. The other step is to alter the Gentzen-type calculi to make it algorithmic. To complete this task, not algorithmic rules (namely, cut) must be removed and termination of derivation search in the calculus must be ensured. Although the dissertation is mainly aimed at multimodal logic $S4_n$ with central agent axiom, during the research similar results for other epistemic logics such as K_n, T_n and $K4_n$ with central agent axiom were obtained and they are also presented here.

More precisely the following tasks have been completed:

1. Basic sound and complete sequent calculi for multimodal logics K_n, T_n, $K4_n$ and $S4_n$ with central agent axiom have been developed.

2. Cut-elimination theorem for the developed calculi has been proved.

3. Finiteness of the derivation search in the developed calculi for multimodal logics K_n and T_n with central agent axiom has been demonstrated with little or no changes to the calculi.

4. New method to obtain termination in derivation search for transitive monomodal logics $K4$ and $S4$ have been developed. New terminating Gentzen-type calculi for these logics have been created.

5. The developed monomodal calculi have been adapted to transitive multimodal logics $K4_n$ and $S4_n$ with central agent axiom and new terminating Gentzen-type calculi for these logics have been obtained.

Methods

The basic Gentzen-type calculi for logics K_n, T_n, $K4_n$ and $S4_n$ with central agent axiom are developed from respective sound and complete Hilbert-type calculi. The soundness and completeness of such calculi is proved by showing the equivalence between Gentzen-type and Hilbert-type calculi. The invertibility of most of the rules of the calculi and admissibility of weakening and contraction structural rules are used to prove the cut-elimination theorem. The finiteness of the derivation search in the developed calculus for logic K_n with central agent axiom is proved by showing that the length of sequent decreases while going up the derivation search tree.

In order to obtain the finiteness of the derivation search in the calculi for other considered logics, new methods, which use labels, have been created.

The idea to use labels dates back to [32]. In this dissertation to restrict applications of reflexivity rule, $*$ is used (similarly to [45]). This is enough for logic T_n with central agent axiom. However, transitivity axiom $\Box_l F \supset \Box_l \Box_l F$ causes a lot of problems in detecting the termination of derivation search in sequent calculi for logics $K4_n$ and $S4_n$ with central agent axiom. Therefore, in addition to the stars, indexes (once again, similar to the ones in [45]) are used to keep track of every occurrence of modality that could lead to a loop. Moreover, marks of indexed occurrences of modality show when transitivity rule was applied to the same formula. Finally formula numbers are used to indicate when the formula appeared in the derivation search tree for the first time. Indexes, marks and formula numbers are used to restrict applications of transitivity rule.

To show the soundness and completeness of the developed terminating calculi, the equivalence between the cut-free and the terminating calculi is proved. The finiteness of the derivation search in the calculi is proved by demonstrating that the value of the ordered multiple decreases while going up the derivation search tree.

Scientific Novelty

In this dissertation new sequent calculi are presented. The mentioned calculi cover multimodal logics K_n, T_n, $K4_n$ and $S4_n$ with central agent axiom and monomodal logics $K4$ and $S4$. There are many terminating calculi for $S4$ (e.g. [22, 29, 37], [45] with corrections in [46]). In [27] a new method of histories is presented, which is widely used in various epistemic logics. However, as mentioned in the article, it is not clear how to extend this method to multimodal logics. What is more, the author is not aware of terminating calculi for multimodal logic $S4_n$ with central agent axiom. Although multimodal calculi with interaction axioms are analysed, they do not cover the mentioned logic. For example, in [21] multimodal logic $KD45_n$ with various interaction axioms is analysed and the terminating calculi are presented. In [23] multimodal logics K_n, T_n and $S4_n$ with distributed knowledge, which is analogous to central agent knowledge as mentioned above, are analysed and cut-free sequent calculi are presented. What is more, it is possible to conclude from the article the cut-free calculus for $K4_n$ with distributed knowledge. However, the mentioned calculi for $K4_n$ and $S4_n$ are not terminating.

The most important result of the dissertation is a new marks and indexes method for transitive monomodal and multimodal logics, in which four kind of labels are used: (1) indexes of the occurrences of modality, (2) marks of indexed modalities, (3) marks $*$ and (4) formula numbers. Although the usage of labels is not a new thing ([32, 45]), in this thesis they are applied in a new and original way. This new method is employed to ensure the termination of every derivation search in sequent calculi for transitive monomodal logics *K4* and *S4* as well as for transitive multimodal logics $K4_n$ and $S4_n$ with central agent axiom.

Finally, the results presented in this dissertation can be easily extended to other multimodal logics. First of all by eliminating the central agent, the terminating calculi for multimodal logics K_n, T_n, $K4_n$ and $S4_n$ are obtained. Moreover, with some effort these calculi can be adapted to the mentioned logics with different interaction axioms. E.g., the system with agents of different capabilities, described in the start of this chapter.

Defending Statements

These statements are presented for defence:

1. New constructed Gentzen-type calculi for multimodal logics K_n, T_n, $K4_n$ and $S4_n$ with central agent axiom demonstrate how central agent axiom can be modelled in sequent calculi without causing problems in cut-elimination.

2. New method of obtaining finite derivation search is developed. This method is applied to obtain new terminating Gentzen-type calculi for transitive monomodal logics *K4* and *S4*.

3. This new method is extended to obtain new terminating Gentzen-type calculi for transitive multimodal logics $K4_n$ and $S4_n$ with central agent axiom. Some ideas of this method are used to construct such calculus for logic T_n with central agent axiom.

Approval of Research Results

The author's research is documented in 6 articles. 2 articles are in the periodical journals, included in Scientific Master Journal List (ISI). Other articles are in the international refereed journals.

The results are also presented in 4 conferences and in the seminars of Mathematical Logic Sector at Software Engineering Department of Vilnius University Institute of Mathematics and Informatics.

Publications of the Author

The publications of the author related to this dissertation are:

1. Scientific articles in the periodical journals, included in Scientific Master Journal List (ISI):
   (a) J. Andrikonis. Cut elimination for S4n and K4n with the central agent axiom. *Lithuanian Mathematical Journal*, **49**(2), pp. 123–139, 2009.
   (b) J. Andrikonis. Loop-free calculus for modal logic S4. *Lithuanian Mathematical Journal*, July 2011. Accepted for publication.
2. Scientific articles in other international refereed journals:
   (a) J. Andrikonis and R. Pliuškevičius. Cut elimination for knowledge logic with interaction. *Lithuanian Mathematical Journal*, **47**(spec. issue), pp. 346–350, 2007.
   (b) J. Andrikonis. Cut-elimination for knowledge logics with interaction. *Lithuanian Mathematical Journal*, **48/49**(spec. issue), pp. 263–268, 2008.
   (c) J. Andrikonis. Loop-free sequent calculus for modal logic K4. *Lithuanian Mathematical Journal*, **50**(spec. issue), pp. 241–246, 2009.
   (d) J. Andrikonis and R. Pliuškevičius. Contraction-free calculi for modal logics S5 and KD45. *Lithuanian Mathematical Journal*, **51**(spec. issue), August 2011. Accepted for publication.

Outline of the Dissertation

In Chapter 1 initial Hilbert-type and Gentzen-type calculi for classical, monomodal and multimodal logics are defined, some important measures are introduced as well as some properties of the mentioned calculi are proved. Among the most important properties are invertibility of most of the rules and admissibility of weakening and contraction in the presented sequent calculi.

In Chapter 2 central agent axiom is presented and the relation between central agent knowledge and distributed knowledge is described. Next Hilbert-type calculi for multimodal logics K_n, T_n, $K4_n$ and $S4_n$ with central agent axiom are presented. From them Gentzen-type calculi are derived and their equivalence is proved. In the final section of the chapter the cut-elimination theorem for all the considered calculi is proved.

Chapter 3 presents terminating sequent calculi for all the considered logics and proves that every derivation search in the presented calculi is finite. In fact, the cut-free calculus for K_n with central agent axiom is already terminating and the one for T_n with central agent axiom requires only minor changes. To make the dissertation clearer, the terminating calculi for monomodal logics $K4$ and $S4$ are presented before discussing the cases of multimodal logics $K4_n$ and $S4_n$ with central agent axiom. The soundness and completeness of all the newly introduced calculi are also proved in the chapter.

Annex A presents the detailed proof of Lemma 2.3.6.

Chapter 1

Initial Calculi

This chapter presents definitions, terms and theorems that are used in the whole dissertation. The results presented in this chapter are not the work of the author and the references are given, where applicable. In other situations, the information presented here is considered to be a common knowledge[1].

1.1 Classical Propositional Calculi

Classical propositional logic is denoted PC. To construct the formulas of PC, standard logical connectives are used: unary operator $\neg$ (negation) and binary operators $\wedge$ (conjunction), $\vee$ (disjunction) and $\supset$ (implication).

Definition 1.1.1. *Classical formula is defined recursively as follows:*

- *Propositional variable is a classical formula.*
- *If F is a classical formula, then $(\neg F)$ is a classical formula too.*
- *If F and G are classical formulas, then $(F \wedge G)$, $(F \vee G)$, $(F \supset G)$ are classical formulas too.*

Formulas are denoted by capital Latin letters $(F, G, H, F_1, \dots)$, propositional variables are denoted by small Latin letters $(p, q, r, p_1, \dots)$. Outermost brackets of formulas in this thesis are always omitted, as well as some inner brackets if the order of application of logical operators is clear. The priorities of logical operators are $\neg$, $\wedge$, $\vee$, $\supset$, where $\neg$ has the highest priority and $\supset$ — the lowest.

[1] As a matter of fact the concept of common knowledge is also analysed in the context of multimodal logic e.g. in [11, 24, 25].

Definition 1.1.2. *An interpretation of classical formula F is function $\nu : \mathcal{P} \to \{\top, \bot\}$, where $\mathcal{P}$ is a set of all the propositional variables of F, $\top$ stands for "true" and $\bot$ stands for "false".*

Now a truth relation $\vDash$ between interpretation and formula is defined.

Definition 1.1.3. *Let's say that ν is an interpretation of classical formula F, then:*

- *if F is a propositional variable, then $\nu \vDash F$ iff $\nu(F) = \top$.*
- *if $F \equiv \neg G$, then $\nu \vDash F$ iff $\nu \nvDash G$.*
- *if $F \equiv G \wedge H$, then $\nu \vDash F$ iff $\nu \vDash G$ and $\nu \vDash H$.*
- *if $F \equiv G \vee H$, then $\nu \nvDash F$ iff $\nu \nvDash G$ and $\nu \nvDash H$.*
- *if $F \equiv G \supset H$, then $\nu \nvDash F$ iff $\nu \vDash G$ and $\nu \nvDash H$.*

If $\nu \vDash F$, then it is said that formula F is true with interpretation ν. If F is true with every possible interpretation of F, then it is said that F is valid and denoted $\vDash F$. Sometimes it is important to clarify the logic, for which the formula is valid (or true with interpretation ν). This is presented as index of truth relation: $\vDash_{PC} F$ (or $\nu \vDash_{PC} F$ respectively).

To test for formula validity several methods are used. Here only Hilbert-type and Gentzen-type calculi are analysed.

In this dissertation Hilbert-type calculus defined in [33] is used. For an alternative definition the reader could refer to [30].

Definition 1.1.4. *Hilbert-type calculus for classical propositional logic (denoted HPC) consists of axioms:*

1.1. $F \supset (G \supset F)$;

1.2. $\big(F \supset (G \supset H)\big) \supset \big((F \supset G) \supset (F \supset H)\big)$;

2.1. $(F \wedge G) \supset F$;

2.2. $(F \wedge G) \supset G$;

2.3. $(F \supset G) \supset \Big((F \supset H) \supset \big(F \supset (G \wedge H)\big)\Big)$;

3.1. $F \supset (F \vee G)$;

3.2. $G \supset (F \vee G)$;

3.3. $(F \supset H) \supset \Big((G \supset H) \supset \big((F \vee G) \supset H\big)\Big)$;

4.1. $(F \supset G) \supset (\neg G \supset \neg F)$;

4.2. $F \supset \neg\neg F$;

4.3. $\neg\neg F \supset F$;

*and Modus Ponens (*MP*) rule:*

$$\frac{F \qquad F \supset G}{G}$$

Here F, G and H stand for any classical formula.

This type of calculus was formulated for the first time in [28], therefore such calculi are called Hilbert-type.

In order to check if some formula is valid, a *derivation* is constructed. A derivation of formula F in Hilbert-type calculus is a sequence of formulas $F_1, \ldots, F_n$, where $F_n \equiv F$ and for every $i \in [1, n]$, F_i is either an axiom, or obtained by applying the rules of the calculus to formulas from the set $\{F_j : j < i\}$. Thus the derivation search starts with axioms and from them new formulas are constructed by applying the rules. The process terminates successfully if formula F is finally obtained.

In this dissertation Hilbert-type derivations are presented as lists together with information on how the formula was obtained. For this purpose an expression of the type $\{^{G_1}/_{F_1}, \ldots, ^{G_n}/_{F_n}\}$, called the *substitution*, is used. This means that in the discussed formula all the occurrences of subformula F_i is replaced by G_i, $\forall i \in [1, n]$.

Definition 1.1.5. *It is said that formula F is derivable in some Hilbert-type calculus $\mathcal{C}$ (denoted $\vdash_{\mathcal{C}} F$), if a derivation of F in $\mathcal{C}$ exists. Otherwise it is said that F is not derivable in $\mathcal{C}$ ($\nvdash_{\mathcal{C}} F$).*

One of the core properties of the calculus are soundness and completeness. It is said that Hilbert-type calculus $\mathcal{C}$ for logic $\mathcal{L}$ is *sound* if for any formula F, if $\vdash_{\mathcal{C}} F$ then $\vDash_{\mathcal{L}} F$. It is said that Hilbert-type calculus $\mathcal{C}$ for logic $\mathcal{L}$ is *complete* if for any formula F, if $\vDash_{\mathcal{L}} F$, then $\vdash_{\mathcal{C}} F$. Only sound and complete calculi can be used to check both validity and invalidity of any formula.

The soundness and completeness of *HPC* is shown in [33].

Now an example of derivation in *HPC* is presented.

Example 1.1.6. *A derivation of $p \supset p$ in HPC is as follows:*

1. $\big(p \supset ((p \supset p) \supset p)\big) \supset \big((p \supset (p \supset p)) \supset (p \supset p)\big)$ Axiom *1.2*, $\{p/F, p\supset p/G, p/H\}$.
2. $p \supset \big((p \supset p) \supset p\big)$ Axiom *1.1*, $\{p/F, p\supset p/G\}$.
3. $\big(p \supset (p \supset p)\big) \supset (p \supset p)$ *MP* rule from 2 and 1.
4. $p \supset (p \supset p)$ Axiom *1.1*, $\{p/F, p/G\}$.
5. $p \supset p$ *MP* rule from 4 and 3.

Although Hilbert-type calculi are used in discussing semantics of the logic, however proof search in such calculi is not an easy task. It is hard to describe an algorithm of choosing the axioms, as can be seen in Example 1.1.6. There are several techniques suggested to tackle this problem, one of witch was first introduced in [18], and therefore is called Gentzen-type calculus. This technique is analysed in the dissertation.

Definition 1.1.7. *A sequent is an expression of the form $\Gamma \to \Delta$, where Γ and Δ are multisets of formulas and can possibly be empty. Γ is called antecedent and Δ is called succedent. The order of the elements of the multisets is not important.*

In this dissertation sequents are denoted by letter S with or without indices and capital Greek letters (Γ, Δ, Σ, Γ_1) denote multisets of formulas, which can be empty, if not mentioned otherwise. Sequents, which consist of classical formulas only, are called classical sequents.

Because of the use of sequents, Gentzen-type calculi are often referred to as the sequent calculi.

Definition 1.1.8. *Let's say that S is a sequent, then corresponding formula of S (denoted $\mathrm{Cor}(S)$) is defined as follows:*

- *if $S = F_1, \ldots, F_n \to G_1, \ldots, G_m$, where $n, m \geqslant 1$, then $\mathrm{Cor}(S) = (F_1 \wedge \ldots \wedge F_n) \supset (G_1 \vee \ldots \vee G_m)$.*
- *if $S = \ \to G_1, \ldots, G_m$, where $m \geqslant 1$, then $\mathrm{Cor}(S) = G_1 \vee \ldots \vee G_m$.*
- *if $S = F_1, \ldots, F_n \to \ $, where $n \geqslant 1$, then $\mathrm{Cor}(S) = \neg(F_1 \wedge \ldots \wedge F_n)$.*
- *if $S = \ \to \ $, then $\mathrm{Cor}(S) = p \wedge \neg p$ for some propositional variable p.*

It is clear that if S is a classical sequent, then $\mathrm{Cor}(S)$ is a classical formula. This definition is similar to that given in [49].

Now it is possible to define the semantic meaning of sequent.

Definition 1.1.9. *Let's say that S is a classical sequent and ν is an interpretation of classical formula $\mathrm{Cor}(S)$, then S is true with interpretation ν*

(denoted $\nu \models S$) iff $\nu \models \mathrm{Cor}(S)$. If $\models \mathrm{Cor}(S)$, then it is said that S is valid and denoted $\models S$.

In this dissertation the sequent calculus provided in [32] is used. This calculus has some very good properties, which will be discussed later. It is also used by other authors, i.e. [13, 20].

Definition 1.1.10. *Gentzen-type calculus for classical propositional logic (GPC) consists of an axiom $\Gamma, F \to F, \Delta$ and the logical rules:*

Negation:

$$\frac{\Gamma \to \Delta, F}{\neg F, \Gamma \to \Delta}\,(\neg \to) \qquad \frac{F, \Gamma \to \Delta}{\Gamma \to \Delta, \neg F}\,(\to \neg)$$

Conjunction:

$$\frac{F, G, \Gamma \to \Delta}{F \wedge G, \Gamma \to \Delta}\,(\wedge \to) \qquad \frac{\Gamma \to \Delta, F \qquad \Gamma \to \Delta, G}{\Gamma \to \Delta, F \wedge G}\,(\to \wedge)$$

Disjunction:

$$\frac{F, \Gamma \to \Delta \qquad G, \Gamma \to \Delta}{F \vee G, \Gamma \to \Delta}\,(\vee \to) \qquad \frac{\Gamma \to \Delta, F, G}{\Gamma \to \Delta, F \vee G}\,(\to \vee)$$

Implication:

$$\frac{\Gamma \to \Delta, F \qquad G, \Gamma \to \Delta}{F \supset G, \Gamma \to \Delta}\,(\supset\to) \qquad \frac{F, \Gamma \to \Delta, G}{\Gamma \to \Delta, F \supset G}\,(\to\supset)$$

The sequent(s) above the horizontal line of the rule is (are) called the *premise(s)*. The sequent bellow the line is called the *conclusion*.

Once again, to check the validity of some sequent, derivation is constructed. Now a *derivation search tree* of the sequent S in Gentzen-type calculus is a tree of sequents, which has S at the bottom as a root and each node is either a leaf, or a conclusion of an application of some rule of the calculus in which case all the premises of the application are child nodes of that node. S is called the initial sequent. To denote some derivation search tree, letter $\mathcal{D}$ with or without indices is used.

A *branch* of derivation search tree $\mathcal{D}$ is a subtree of $\mathcal{D}$ in which each node except the last one has exactly one child, the last node has no children and which is not a subtree of any other branch of $\mathcal{D}$ (similar definition can be found in [52]). The branch can be infinite. In that case there is no last node and each node has exactly one child. It can be noticed that every sequent of the derivation search tree belongs to at least one branch. If every branch of a derivation search tree $\mathcal{D}$ is finite, then $\mathcal{D}$ is *finite*, otherwise it is *infinite*. A part of a branch between sequents S_1 and S_2 is called a *path* from S_1 to S_2.

If all the branches of a derivation search tree $\mathcal{D}$ of S end with axiom, it is said that $\mathcal{D}$ is a *derivation tree* (or simply a derivation) of S. If there exists a derivation tree of S in Gentzen-type calculus $\mathcal{C}$, it is said that S is *derivable* in $\mathcal{C}$ (denoted $\vdash_{\mathcal{C}} S$) and otherwise it is said that S is *not derivable* (denoted $\nvdash_{\mathcal{C}} S$). Formula F is derivable in sequent calculus $\mathcal{C}$ (denoted $\vdash_{\mathcal{C}} F$) iff $\vdash_{\mathcal{C}} \rightarrow F$.

Similarly to the Hilbert-type calculus, if the derivation tree of sequent S in sequent calculus is present, the reasoning about the validity of S is obvious. It starts from the axiom(s) and is continued through the applications of the rules. Due to the form of the rules, if all the premises of some application are valid, the conclusion of the application is valid too. However, the process of derivation search starts with sequent S. If S is not an axiom and it is suitable as a conclusion of some rule, then the premise(s) of the rule are inspected and the process of finding the appropriate rule is repeated to them. Thus in this dissertation it is said that the rule of sequent calculus is applied to the conclusion and the premise(s) are obtained. The application of some rule is called an *inference*.

It is obvious, that usually several rules can be applied to the same sequent. To separate them, *main formula* of the inference is defined. The main formula of inferences

$$\frac{\Gamma \rightarrow \Delta, F}{\neg F, \Gamma \rightarrow \Delta}\,(\neg \rightarrow) \qquad \frac{F, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg F}\,(\rightarrow \neg)$$

is $\neg F$. F is called a *side formula*. The main formula of inferences

$$\frac{F, G, \Gamma \rightarrow \Delta}{F \wedge G, \Gamma \rightarrow \Delta}\,(\wedge \rightarrow) \qquad \frac{\Gamma \rightarrow \Delta, F \qquad \Gamma \rightarrow \Delta, G}{\Gamma \rightarrow \Delta, F \wedge G}\,(\rightarrow \wedge)$$

is $F \wedge G$. F and G are side formulas. The main formula of inferences

$$\frac{F, \Gamma \rightarrow \Delta \qquad G, \Gamma \rightarrow \Delta}{F \vee G, \Gamma \rightarrow \Delta}\,(\vee \rightarrow) \qquad \frac{\Gamma \rightarrow \Delta, F, G}{\Gamma \rightarrow \Delta, F \vee G}\,(\rightarrow \vee)$$

is $F \vee G$. F and G are side formulas. The main formula of inferences

$$\frac{\Gamma \rightarrow \Delta, F \qquad G, \Gamma \rightarrow \Delta}{F \supset G, \Gamma \rightarrow \Delta}\,(\supset\rightarrow) \qquad \frac{F, \Gamma \rightarrow \Delta, G}{\Gamma \rightarrow \Delta, F \supset G}\,(\rightarrow\supset)$$

is $F \supset G$. F and G are side formulas. The main formula of axiom $\Gamma, F \rightarrow F, \Delta$ is formula F.

It is said that sequent calculus $\mathcal{C}$ for logic $\mathcal{L}$ is *sound* if for any sequent S, if $\vdash_{\mathcal{C}} S$ then $\vDash_{\mathcal{L}} S$. It is said that sequent calculus $\mathcal{C}$ for logic $\mathcal{L}$ is *complete* if for any sequent S, if $\vDash_{\mathcal{L}} S$, then $\vdash_{\mathcal{C}} S$. In [32] it is proved that GPC is sound and complete.

Now an example of derivation in GPC is given.

Example 1.1.11. *A derivation in GPC of the formula used in Example 1.1.6 is obvious, so a derivation tree of Axiom* 2.3 *of HPC is provided instead.*

$$\dfrac{\dfrac{\dfrac{\dfrac{F, F \supset G \rightarrow G \wedge H, F \qquad \dfrac{F, H \rightarrow G \wedge H, F \qquad \dfrac{F, H, G \rightarrow G \qquad F, H, G \rightarrow H}{F, H, G \rightarrow G \wedge H}\,(\wedge\rightarrow)}{F, H, F \supset G \rightarrow G \wedge H}\,(\supset\rightarrow)}{F, F \supset H, F \supset G \rightarrow G \wedge H}\,(\supset\rightarrow)}{F \supset H, F \supset G \rightarrow F \supset (G \wedge H)}\,(\rightarrow\supset)}{F \supset G \rightarrow (F \supset H) \supset \big(F \supset (G \wedge H)\big)}\,(\rightarrow\supset)}{\rightarrow (F \supset G) \supset \Big((F \supset H) \supset \big(F \supset (G \wedge H)\big)\Big)}\,(\rightarrow\supset)$$

1.2 Modal Calculi

In order to use logic to reason about modalities such as knowledge, belief, obligation two modal logical operators are introduced: $\Box$ (necessity) and $\Diamond$ (possibility). However in this dissertation only necessity modality is analysed, because possibility can be replaced by $\Box$ modality with equivalence $\Diamond F \equiv \neg\Box\neg F$ as explained latter. In epistemic logic modality $\Box$ is interpreted as knowledge operator.

Definition 1.2.1. *Modal formula is defined in following recursive way:*

- *Propositional variable is a modal formula.*
- *If F is a modal formula, then $(\neg F)$ and $(\Box F)$ are modal formulas too.*
- *If F and G are modal formulas, then $(F \wedge G)$, $(F \vee G)$, $(F \supset G)$ are modal formulas too.*

The operator $\Box$ together with $\neg$ has the highest priority in determining the order of application of logical operators in formula. A sequent that contains only modal formulas, is called modal sequent. Modal formulas, modal sequents and multisets, that contain modal formulas, are denoted in the same way as the classical ones.

To define, which formula is true in modal logic, Kripke structure (introduced in [34]) is used.

Definition 1.2.2. *A Kripke structure for modal formula F is a triple $\langle \mathcal{W}, \mathcal{R}, \Phi \rangle$, where*

1. *$\mathcal{W}$ is a set of worlds,*
2. *$\mathcal{R}$ is a binary relation between the elements of $\mathcal{W}$,*
3. *$\Phi : \mathcal{W} \times \mathcal{P} \to \{\top, \bot\}$ is interpretation function, where $\mathcal{P}$ is a set of all the propositional variables of F.*

A pair $\langle \mathcal{W}, \mathcal{R} \rangle$ of Kripke structure $\mathcal{S} = \langle \mathcal{W}, \mathcal{R}, \Phi \rangle$ is called a *frame.*

Kripke structure contains the set of worlds and depending on the interpretation function a propositional variable can get different value in different world. Therefore, modal formula can be true in one world of the structure and false in another one. An expression $\mathcal{S}, w \vDash F$ denotes that formula F is true in world w of Kripke structure $\mathcal{S}$.

Now the truth relation is defined as follows.

Definition 1.2.3. *Let's say that $\mathcal{S} = \langle \mathcal{W}, \mathcal{R}, \Phi \rangle$ is a Kripke structure for modal formula F and $w \in \mathcal{W}$ is some world, then:*

- *if F is a propositional variable, then $\mathcal{S}, w \vDash F$ iff $\Phi(w, F) = \top$.*
- *if $F \equiv \neg G$, then $\mathcal{S}, w \vDash F$ iff $\mathcal{S}, w \nvDash G$.*
- *if $F \equiv G \wedge H$, then $\mathcal{S}, w \vDash F$ iff $\mathcal{S}, w \vDash G$ and $\mathcal{S}, w \vDash H$.*
- *if $F \equiv G \vee H$, then $\mathcal{S}, w \nvDash F$ iff $\mathcal{S}, w \nvDash G$ and $\mathcal{S}, w \nvDash H$.*
- *if $F \equiv G \supset H$, then $\mathcal{S}, w \nvDash F$ iff $\mathcal{S}, w \vDash G$ and $\mathcal{S}, w \nvDash H$.*
- *if $F \equiv \Box G$, then $\mathcal{S}, w \vDash F$ iff $\forall w_1, w\mathcal{R}w_1 : \mathcal{S}, w_1 \vDash G$.*[2]

If formula F is true in every world of Kripke structure $\mathcal{S}$, then it is said, that formula F is valid in $\mathcal{S}$ and denoted $\mathcal{S} \vDash F$.

The weakest modal logic is K, which adds only one axiom to *HPC*. However, usually some additional properties of $\Box$ must be defined and therefore,

[2]Posibility modality is defined in a following way: if $F \equiv \Diamond G$, then $\mathcal{S}, w \vDash F$ iff $\exists w_1, w\mathcal{R}w_1 : \mathcal{S}, w_1 \vDash G$. It is clear, that $\mathcal{S}, w \vDash \Diamond G$ iff $\mathcal{S}, w \vDash \neg\Box\neg G$ and therefore all the occurrences of $\Diamond$ can be replaced by $\neg\Box\neg$.

the calculus is extended with additional axioms. The notation $\mathcal{L}_1 = \mathcal{L} + (\mathcal{A})$ is used to define that modal operator $\Box$ in logic $\mathcal{L}_1$ posses the same properties as the one in logic $\mathcal{L}$ and a property defined by axiom $(\mathcal{A})$. Analogously, if $\mathcal{C}$ is a Hilbert-type calculus, then the notation $\mathcal{C}_1 = \mathcal{C} + (\mathcal{A})$ means that Hilbert-type calculus $\mathcal{C}_1$ is obtained from calculus $\mathcal{C}$ by adding axiom $(\mathcal{A})$.

The following formulation of calculus is traditional (see [15, 49]):

Definition 1.2.4. *Hilbert-type calculus for modal logic K (HK) consists of the same axioms as HPC, axiom* (K)*:*

$$\Box(F \supset G) \supset (\Box F \supset \Box G)$$

MP *rule and Necessity Generalization rule (*NG*):*

$$\frac{F}{\Box F}$$

Other axioms for modal logics are:

(T)*:* $\Box F \supset F$

(4)*:* $\Box F \supset \Box\Box F$

Other modal logics are defined as follows: $T = K + (\mathrm{T})$*,* $K4 = K + (4)$*,* $S4 = T + (4) = K4 + (\mathrm{T})$*.*

Hilbert type calculi for respective modal logics are defined analogously: $HT = HK + (\mathrm{T})$*,* $HK4 = HK + (4)$*,* $HS4 = HT + (4) = HK4 + (\mathrm{T})$*.*

In all the defined modal calculi F, G and H stand for any modal formula. This applies to the newly introduced axioms and rules as well as to the ones, which are inherited from HPC.

Axiom *(T)* is usually called the knowledge axiom or the truth axiom ([11]). It states, that everything, that is known, is true and in general is used to distinguish knowledge from belief. Axiom *(4)* is called the positive introspection and according to it the agent knows, what it knows. Logic *S5* adds another axiom to the ones of *S4*: $\neg\Box F \supset \Box\neg\Box F$. This axiom is called the negative introspection (or *(5)*) and it says that agent knows, what it doesn't know. If the knowledge of humans is discussed, it is very likely that this axiom doesn't hold ([41]).

Now let's analyse an example of Kripke structure and how it is decided if the formula is true.

Example 1.2.5. *Let's say that* $\mathcal{S} = \langle \mathcal{W}, \mathcal{R}, \Phi \rangle$ *is a Kripke structure for formula* $F = \Box p \supset p$, *where* $\mathcal{W} = \{w_1, w_2\}$, $\mathcal{R} = \{(w_1, w_2)\}$, $\Phi(p, w_1) = \bot$ *and* $\Phi(p, w_2) = \top$. *It is possible to demonstrate the structure graphically:*

$$\begin{tikzcd} \overset{p=\bot}{\underset{w_1}{\bullet}} \arrow[rr] & & \overset{p=\top}{\underset{w_2}{\bullet}} \end{tikzcd}$$

Now let's check, if $\mathcal{S}, w_1 \vDash F$. *According to the definition* $\mathcal{S}, w_1 \vDash \Box p$, *because* $\mathcal{S}, w_2 \vDash p$ *and* w_2 *is the only world such that* $w_1 \mathcal{R} w_2$. *However* $\mathcal{S}, w_1 \nvDash p$. *Therefore* $\mathcal{S}, w_1 \nvDash \Box p \supset p$.

This example demonstrates, that it is possible to construct a Kripke structure, in which axiom *(T)* is not valid. Indeed, for this axiom to be valid in each world of Kripke structure $\mathcal{S} = \langle \mathcal{W}, \mathcal{R}, \Phi \rangle$, the frame $\langle \mathcal{W}, \mathcal{R} \rangle$ of $\mathcal{S}$ must be reflexive: $\forall w \in \mathcal{W} : w \mathcal{R} w$. Similarly, axiom *(4)* requires the frame to be transitive ($\forall w_1, w_2, w_3 \in \mathcal{W}$: if $w_1 \mathcal{R} w_2$ and $w_2 \mathcal{R} w_3$, then $w_1 \mathcal{R} w_3$). Therefore, the validity of formula in modal logic is defined taking into account the requirements of the axioms, that are part of the logic:

- $\vDash_K F$ iff for any Kripke structure $\mathcal{S}$ it is true that $\mathcal{S} \vDash F$.
- $\vDash_T F$ iff for any Kripke structure $\mathcal{S}$ with reflexive frame $\mathcal{S} \vDash F$.
- $\vDash_{K4} F$ iff for any Kripke structure $\mathcal{S}$ with transitive frame $\mathcal{S} \vDash F$.
- $\vDash_{S4} F$ iff for any Kripke structure $\mathcal{S}$ with reflexive and transitive frame $\mathcal{S} \vDash F$.

Because of that, *(T)* is called reflexivity axiom and *(4)* — transitivity axiom.

For some modal sequent S, Kripke structure $\mathcal{S} = \langle \mathcal{W}, \mathcal{R}, \Phi \rangle$ for modal formula $\mathrm{Cor}(S)$ and world $w \in \mathcal{W}$ it is said that (1) $\mathcal{S}, w \vDash S$ iff $\mathcal{S}, w \vDash \mathrm{Cor}(S)$, (2) $\mathcal{S} \vDash S$ iff $\mathcal{S} \vDash \mathrm{Cor}(S)$ and (3) $\vDash_{\mathcal{L}} S$, iff $\vDash_{\mathcal{L}} \mathrm{Cor}(S)$ for some logic $\mathcal{L}$.

It is known that calculi *HK*, *HT*, *HK4* and *HS4* are sound and complete.

The definition of sequent calculi, used here, are based on the ones provided in [36, 48] (for *K*), [16, 48] (for *K4*) and [44] (for *T* and *S4*). Alternative formulations can be found in [19, 53].

Definition 1.2.6. *Gentzen-type calculus for modal logics consists of an axiom $\Gamma, F \to F, \Delta$, the logical rules of GPC and modal rules, which depend on logic:*

Calculus GK for modal logic K:

$$\frac{\Gamma_2 \to F}{\Gamma_1, \Box\Gamma_2 \to \Delta, \Box F} \; (\to \Box)$$

Calculus GK4 for modal logic K4:

$$\frac{\Gamma_2, \Box\Gamma_2 \to F}{\Gamma_1, \Box\Gamma_2 \to \Delta, \Box F} \; (\to \Box)$$

Calculus GT for modal logic T:

$$\frac{F, \Box F, \Gamma \to \Delta}{\Box F, \Gamma \to \Delta} \; (\Box \to) \qquad \frac{\Gamma_2 \to F}{\Gamma_1, \Box\Gamma_2 \to \Delta, \Box F} \; (\to \Box)$$

Calculus GS4 for modal logic S4:

$$\frac{F, \Box F, \Gamma \to \Delta}{\Box F, \Gamma \to \Delta} \; (\Box \to) \qquad \frac{\Box\Gamma_2 \to F}{\Gamma_1, \Box\Gamma_2 \to \Delta, \Box F} \; (\to \Box)$$

In all the new rules $\Box F$ is main formula and F is side formula. The rule $(\Box \to)$ is called *reflexivity* ([45]), because it corresponds to reflexivity axiom of Hilbert-type calculus for *T* and *S4* as can be seen in proof of Lemma 2.3.11. The rule $(\to \Box)$ of *GK4* and *GS4* is called *transitivity*, because it corresponds to transitivity axiom of *HK4* and *HS4* as can be seen in proofs of Lemmas 2.3.10 and 2.3.12.

Calculi *GK*, *GK4*, *GT* and *GS4* are sound and complete.

1.3 Multimodal Calculi

To reason about knowledge of many agents, multimodal logic is used. In such case knowledge of agent l is denoted as $\Box_l$. Agents are usually numbered with natural numbers starting from 1, however special agents may get a different name. A set of agents may be finite as well as infinite.

Definition 1.3.1. *Multimodal formula is defined in following recursive way:*

- *Propositional variable is a multimodal formula.*
- *If F is a multimodal formula, then $(\neg F)$ and $(\Box_l F)$, where l is a name of agent, are multimodal formulas too.*

- *If F and G are multimodal formulas, then $(F \wedge G)$, $(F \vee G)$, $(F \supset G)$ are multimodal formulas too.*

The operator $\Box_l$ together with $\neg$ has the highest priority in determining the order of application of logical operators in formula. A sequent that contains only multimodal formulas, is called multimodal sequent. Multimodal formulas, sequents and multisets, that contain multimodal formulas, are denoted in the same way as the classical and modal ones.

To define the validity of multimodal formulas, Kripke structure must also be changed.

Definition 1.3.2. *A Kripke structure for multimodal formula F is a multiple $\langle \mathcal{W}, \mathcal{R}_{l_1}, \dots, \mathcal{R}_{l_n}, \Phi \rangle$, where*

1. *$\mathcal{W}$ is a set of worlds,*
2. *For each different agent l_j in formula F, $\mathcal{R}_{l_j}$ is a binary relation between the elements of $\mathcal{W}$,*
3. *$\Phi : \mathcal{W} \times \mathcal{P} \to \{\top, \bot\}$ is interpretation function, where $\mathcal{P}$ is a set of all the propositional variables of F.*

Multiple $\langle \mathcal{W}, \mathcal{R}_{l_1}, \dots, \mathcal{R}_{l_n} \rangle$ of Kripke structure $\mathcal{S} = \langle \mathcal{W}, \mathcal{R}_{l_1}, \dots, \mathcal{R}_{l_n}, \Phi \rangle$ is called a frame. It is said that frame $\langle \mathcal{W}, \mathcal{R}_{l_1}, \dots, \mathcal{R}_{l_n} \rangle$ is reflexive (transitive) if every relation $\mathcal{R}_{l_j}, j \in [1, n]$ is reflexive (respectively transitive).

The truth relation for multimodal formulas is defined as follows.

Definition 1.3.3. *Let's say that $\mathcal{S} = \langle \mathcal{W}, \mathcal{R}_{l_1}, \dots, \mathcal{R}_{l_n}, \Phi \rangle$ is a Kripke structure for formula F and $w \in \mathcal{W}$ is some world, then:*

- *if F is a propositional variable, $F \equiv \neg G$, $F \equiv G \wedge H$, $F \equiv G \vee H$ or $F \equiv G \supset H$, then $\mathcal{S}, w \vDash F$ is defined in the same way as in Definition 1.2.3.*
- *if $F \equiv \Box_{l_j} G, j \in [1, n]$, then $\mathcal{S}, w \vDash F$ iff $\forall w_1, w\mathcal{R}_{l_j} w_1 : \mathcal{S}, w_1 \vDash G$.*

Once again if multimodal formula F is true in every world of Kripke structure $\mathcal{S}$, then it is said, that F is valid in $\mathcal{S}$ and denoted $\mathcal{S} \vDash F$. The definition of validity of multimodal formula also takes into account the requirements of the axioms:

- $\vDash_{K_n} F$ iff for any Kripke structure $\mathcal{S}$ it is true that $\mathcal{S} \vDash F$.
- $\vDash_{T_n} F$ iff for any Kripke structure $\mathcal{S}$ with reflexive frame $\mathcal{S} \vDash F$.

- $\vDash_{K4_n} F$ iff for any Kripke structure $\mathcal{S}$ with transitive frame $\mathcal{S} \vDash F$.
- $\vDash_{S4_n} F$ iff for any Kripke structure $\mathcal{S}$ with reflexive and transitive frame $\mathcal{S} \vDash F$.

Once again, for some multimodal sequent S, some Kripke structure $\mathcal{S} = \langle \mathcal{W}, \mathcal{R}_{l_1}, \ldots, \mathcal{R}_{l_n}, \Phi \rangle$ for multimodal formula $\mathrm{Cor}(S)$ and world $w \in \mathcal{W}$ it is said that (1) $\mathcal{S}, w \vDash S$ iff $\mathcal{S}, w \vDash \mathrm{Cor}(S)$, (2) $\mathcal{S} \vDash S$ iff $\mathcal{S} \vDash \mathrm{Cor}(S)$ and (3) $\vDash_{\mathcal{L}} S$, iff $\vDash_{\mathcal{L}} \mathrm{Cor}(S)$ for some logic $\mathcal{L}$.

Hilbert-type calculi for multimodal logics are defined in the similar way.

Definition 1.3.4. *Hilbert-type calculus for multimodal logic K_n (HK_n) consists of the same axioms as HPC, axiom* (K_l)*:*

$$\Box_l(F \supset G) \supset (\Box_l F \supset \Box_l G)$$

MP *rule and Necessity Generalization rule (*NG_l*):*

$$\frac{F}{\Box_l F}$$

Other axioms for multimodal logics are:

(T_l)*:* $\Box_l F \supset F$

(4_l)*:* $\Box_l F \supset \Box_l \Box_l F$

Other multimodal logics are defined as follows: $T_n = K_n + (\mathrm{T}_l)$*,* $K4_n = K_n + (4_l)$*,* $S4_n = T_n + (4_l) = K4_n + (\mathrm{T}_l)$*.*

Hilbert type calculi for respective multimodal logics are defined analogously: $HT_n = HK_n + (\mathrm{T}_l)$*,* $HK4_n = HK_n + (4_l)$*,* $HS4_n = HT_n + (4_l) = HK4 + (\mathrm{T}_l)$*.*

Once again in all the defined multimodal calculi F, G and H stand for any multimodal formula.

Similarly to the monomodal variant, *(T_l)* is called reflexivity axiom and *(4_l)* — transitivity axiom. If however agent l must be mentioned, then they are called respectively l-reflexivity and l-transitivity axioms.

It is known that calculi HK_n, HT_n, $HK4_n$ and $HS4_n$ are sound and complete.

The definition of sequent calculi for multimodal logics are also similar.

Definition 1.3.5. *Gentzen-type calculus for multimodal logics consists of an axiom* $\Gamma, F \to F, \Delta$*, the logical rules of GPC and modal rules, which depend on logic:*

Calculus GK_n *for modal logic* K_n*:*

$$\frac{\Gamma_2 \to F}{\Gamma_1, \Box_l \Gamma_2 \to \Delta, \Box_l F} \; (\to \Box_l)$$

Calculus $GK4_n$ *for modal logic* $K4_n$*:*

$$\frac{\Gamma_2, \Box_l \Gamma_2 \to F}{\Gamma_1, \Box_l \Gamma_2 \to \Delta, \Box_l F} \; (\to \Box_l)$$

Calculus GT_n *for modal logic* T_n*:*

$$\frac{F, \Box_l F, \Gamma \to \Delta}{\Box_l F, \Gamma \to \Delta} \; (\Box_l \to) \qquad \frac{\Gamma_2 \to F}{\Gamma_1, \Box_l \Gamma_2 \to \Delta, \Box_l F} \; (\to \Box_l)$$

Calculus $GS4_n$ *for modal logic* $S4_n$*:*

$$\frac{F, \Box_l F, \Gamma \to \Delta}{\Box_l F, \Gamma \to \Delta} \; (\Box_l \to) \qquad \frac{\Box_l \Gamma_2 \to F}{\Gamma_1, \Box_l \Gamma_2 \to \Delta, \Box_l F} \; (\to \Box_l)$$

In all the new rules $\Box_l F$ is main formula and F is side formula. Once again the rule $(\Box_l \to)$ is called *reflexivity* and the rule $(\to \Box_l)$ of $GK4_n$ and $GS4_n$ is called *transitivity*. Similarly, if it is important to mention specific agent l, then they are called respectively l-reflexivity and l-transitivity rules.

Calculi GK_n, $GK4_n$, GT_n and $GS4_n$ are sound and complete.

1.4 Some Properties of the Calculi

First, let's define the height of the derivation. It is the core property and is often used in proofs. The height of the derivation $\mathcal{D}$ is denoted $\mathrm{h}(\mathcal{D})$, however the definition depends on the type of the calculus.

Definition 1.4.1. *A height of a derivation* $F_1, \ldots, F_n$ *in Hilbert-type calculus is:*

- 1*, if* F_n *is an axiom;*
- $h + 1$*, if* F_n *is obtained from* $F_i, i \in [1, n)$ *by applying a rule of the calculus and* $h = \mathrm{h}(F_1, \ldots, F_i)$*;*
- $\max(h_1, h_2) + 1$*, if* F_n *is obtained from* F_i *and* $F_j, i, j \in [1, n)$ *by applying a rule of the calculus,* $h_1 = \mathrm{h}(F_1, \ldots, F_i)$ *and* $h_2 = \mathrm{h}(F_1, \ldots, F_j)$*.*

Definition 1.4.2. *A height of a finite derivation search tree $\mathcal{D}$ in sequent calculus is:*

- 1, *if $\mathcal{D}$ consists of single sequent only;*
- $h+1$, *if the last application of the rule in $\mathcal{D}$ is*

$$\frac{S_1}{S}$$

  and the height of derivation search tree of S_1 is h;
- $\max(h_1, h_2)+1$, *if the last application of the rule in $\mathcal{D}$ is*

$$\frac{S_1 \qquad S_2}{S}$$

  the height of derivation search tree of S_1 is h_1 and the height of derivation search tree of S_2 is h_2.

Another important measure is the length of the formula.

Definition 1.4.3. *The length of formula F (denoted $\mathrm{l}(F)$) is:*

- 0, *if F is a propositional variable.*
- $l+1$, *if F is of the form $\neg G$, $\Box G$ or $\Box_l G$ and $l = \mathrm{l}(G)$.*
- $l_1 + l_2 + 1$, *if F is of the form $G \wedge H$, $G \vee H$ or $G \supset H$, $l_1 = \mathrm{l}(G)$ and $l_2 = \mathrm{l}(H)$.*

The length of multiset of formulas $\Gamma = F_1, \ldots, F_n$ (denoted $\mathrm{l}(\Gamma)$) is equal to $\sum_{i=1}^{n} \mathrm{l}(F_i)$. The length of sequent $S = \Gamma \to \Delta$ (denoted $\mathrm{l}(S)$) is equal to $\mathrm{l}(\Gamma) + \mathrm{l}(\Delta)$.

Definition 1.4.4. *In sequent $F_1, \ldots, F_n \to G_1, \ldots, G_m$ formulas $F_i, i \in [1, n]$ occur negatively and formulas $G_j, j \in [1, m]$ occur positively. If formula or subformula F in the sequent occurs positively (negatively) and*

- *$F = \Box G$ or $F = \Box_l G$, then G occurs in the sequent positively (respectively negatively).*
- *$F = \neg G$, then G occurs in the sequent negatively (respectively positively).*
- *$F = G \wedge H$ or $F = G \vee H$, then G and H occur in the sequent positively (respectively negatively).*

- $F = G \supset H$*, then* G *occurs in the sequent negatively (respectively positively) and* H *occurs in the sequent positively (respectively negatively).*

If formula or subformula $\Box_l F$ *(or* $\Box F$*) occurs in the sequent positively (negatively), then it is said that this occurrence of* $\Box_l$ *(respectively* $\Box$*) is positive (respectively negative).*

It should be noted, that usually in derivation search trees of sequent calculi the occurrences of formula do not change their positiveness. This is true for all the Gentzen-type calculi analysed in this thesis. Thus for example in $GS4_n$ positive occurrence of $\Box_l F$ can only be the main formula of l-transitivity rule and negative occurrence of $\Box_l F$ can only be the main formula of l-reflexivity rule.

Sometimes to ease the proofs of the theorems some additional rules are incorporated in the calculus. If the inclusion of the rule does not alter the set of formulas that are derivable in the calculus, then the rule is called admissible in the calculus. More formally:

Definition 1.4.5. *Let* $\mathcal{C}$ *be some calculus and* $(\mathcal{R})$ *some rule. The rule* $(\mathcal{R})$ *is admissible in calculus* $\mathcal{C}$ *if for every formula* F *it is true that* $\vdash_{\mathcal{C}} F$ *iff* $\vdash_{\mathcal{C}+(\mathcal{R})} F$*.*

It is usually obvious, that if some formula is derivable in calculus $\mathcal{C}$, then it is derivable in $\mathcal{C} + (\mathcal{R})$. Therefore only the fact, that all the applications of the rule $(\mathcal{R})$ can be eliminated from the derivations in $\mathcal{C} + (\mathcal{R})$ must be proved. Admissible rules are convenient, because they can be used in the derivation search without breaking soundness and completeness of the calculus.

Now admissibility of some rules is proved, because these rules are used later.

Lemma 1.4.6. *The rules of negation (*$\mathrm{R}\neg$*), disjunction (*$\mathrm{R}\vee$*), conjunction (*$\mathrm{R}\wedge$*), implication (*$\mathrm{R}\supset$*), transitivity (*Tr*) and substitution in disjunction (*SD*):*

$$\frac{F \supset G}{\neg G \supset \neg F}\,\mathrm{R}\neg \qquad \frac{F \supset H \qquad G \supset H}{(F \vee G) \supset H}\,\mathrm{R}\vee \qquad \frac{F \supset G \qquad F \supset H}{F \supset (G \wedge H)}\,\mathrm{R}\wedge$$

$$\frac{F \supset (G \supset H) \qquad F \supset G}{F \supset H}\,\mathrm{R}\supset \qquad \frac{F \supset G \qquad G \supset H}{F \supset H}\,\mathrm{Tr}$$

$$\frac{F \vee G \quad F \supset H}{H \vee G}\,\mathrm{SD}$$

are admissible in all the defined Hilbert-type calculi[3]*.*

Proof. To prove the admissibility, let's change every application of the considered rule by the following fragments.

In the case of negation rule:

1.	$F \supset G$	An assumption of the rule.
2.	$(F \supset G) \supset (\neg G \supset \neg F)$	Axiom *4.1*.
3.	$\neg G \supset \neg F$	*MP* rule from 1 and 2.

In the case of disjunction rule:

1.	$F \supset H$	An assumption of the rule.
2.	$G \supset H$	An assumption of the rule.
3.	$(F \supset H) \supset \big((G \supset H) \supset ((F \vee G) \supset H)\big)$	Axiom *3.3*.
4.	$(G \supset H) \supset \big((F \vee G) \supset H\big)$	*MP* rule from 1 and 3.
5.	$(F \vee G) \supset H$	*MP* rule from 2 and 4.

In the case of conjunction rule:

1.	$F \supset G$	An assumption of the rule.
2.	$F \supset H$	An assumption of the rule.
3.	$(F \supset G) \supset \big((F \supset H) \supset (F \supset (G \wedge H))\big)$	Axiom *2.3*.
4.	$(F \supset H) \supset \big(F \supset (G \wedge H)\big)$	*MP* rule from 1 and 3.
5.	$F \supset (G \wedge H)$	*MP* rule from 2 and 4.

In the case of implication rule:

1.	$F \supset (G \supset H)$	An assumption of the rule.
2.	$F \supset G$	An assumption of the rule.
3.	$\big(F \supset (G \supset H)\big) \supset \big((F \supset G) \supset (F \supset H)\big)$	Axiom *1.2*.
4.	$(F \supset G) \supset (F \supset H)$	*MP* rule from 1 and 3.
5.	$F \supset H$	*MP* rule from 2 and 4.

In the case of transitivity rule:

1.	$F \supset G$	An assumption of the rule.
2.	$G \supset H$	An assumption of the rule.
3.	$(G \supset H) \supset \big(F \supset (G \supset H)\big)$	Axiom *1.1*, $\{{}^{G \supset H}/_{F}, {}^{F}/_{G}\}$.
4.	$F \supset (G \supset H)$	*MP* rule from 2 and 3.
5.	$F \supset H$	$R\supset$ rule from 4 and 1.

[3]Namely, *HPC*, *HK*, *HK4*, *HT*, *HS4*, HK_n, $HK4_n$, HT_n and $HS4_n$.

In the case of substitution in disjunction rule:

1. $F \vee G$ An assumption of the rule.
2. $F \supset H$ An assumption of the rule.
3. $H \supset (H \vee G)$ Axiom *3.1*, $\{^H/_F, ^G/_G\}$.
4. $F \supset (H \vee G)$ *Tr* rule from 2 and 3.
5. $G \supset (H \vee G)$ Axiom *3.2*, $\{^H/_F, ^G/_G\}$.
6. $(F \vee G) \supset (H \vee G)$ $R\vee$ rule from 4 and 5.
7. $H \vee G$ *MP* rule from 1 and 6.

$\square$

Lemma 1.4.7. *The* K_l *rule:* $\dfrac{\square_l(F \supset G)}{\square_l F \supset \square_l G}$ *is admissible in all the defined multimodal Hilbert-type calculi.*

Proof. Once again, all the applications of this rule can be replaced by following fragment:

1. $\square_l(F \supset G)$ An assumption of the rule.
2. $\square_l(F \supset G) \supset (\square_l F \supset \square_l G)$ Axiom *(K_l)*.
3. $\square_l F \supset \square_l G$ *MP* rule from 1 and 2.

$\square$

It should be noticed, that the proof of Lemma 1.4.6 uses only *HPC* axioms and rules and the proof of Lemma 1.4.7 additionally uses axiom *(K_l)*. Of course, this statement can be formulated in more specific way, however this is enough to ensure, that the considered rules are admissible in any Hilbert-type calculus, that contains axioms and rules of *HPC* and axiom *(K_l)*.

Lemma 1.4.8. *The structural rules of weakening:*

$$\frac{\Gamma \to \Delta}{F, \Gamma \to \Delta}\,(w \to) \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, F}\,(\to w)$$

are admissible in all the defined Gentzen-type calculi[4]*. The main formula of these rules is* F*.*

Proof. Once again it is enough to show, that it is possible to eliminate the weakening structural rules from the derivation trees, however this time the proof depends on the calculus. Let's analyse only the rule $(w \to)$ and only the case of GK_n. Other cases are analogous.

To prove the lemma it is enough to analyse only the derivation trees, which have only one application of the rule $(w \to)$, which is the bottom-most inference in the derivation tree. It is enough to show that such application

[4]Namely, *GPC*, *GK*, *GK4*, *GT*, *GS4*, GK_n, $GK4_n$, GT_n and $GS4_n$.

can be eliminated. After that the discussion can be extended to every derivation tree by induction on the number of applications of $(w \rightarrow)$ rule.

Let $\mathcal{D}$ be such derivation of sequent $S = F, \Gamma \rightarrow \Delta$ and $S_1 = \Gamma \rightarrow \Delta$ be the presumption of the bottom-most application of $(w \rightarrow)$. The proof is by induction on $\mathrm{h}(\mathcal{D})$. If $\mathrm{h}(\mathcal{D}) = 2$, then $\mathcal{D}$ is of the form:

$$\dfrac{S_1 = G, \Gamma_1 \rightarrow \Delta_1, G}{S = F, G, \Gamma_1 \rightarrow \Delta_1, G}\,(w \rightarrow)$$

It is obvious, that in this case S is axiom of the calculus and the application of $(w \rightarrow)$ is not needed.

Let's say that the application of $(w \rightarrow)$ can be eliminated from the derivation tree, if the height of it is less than h. Let $\mathrm{h}(\mathcal{D}) = h$ and let's check what rule is applied to S_1 in $\mathcal{D}$. All the cases of logical rules are similar. Let's analyse only $(\neg \rightarrow)$. In that case $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_2\\ S_2 = \Gamma_1 \rightarrow \Delta, G\end{array}}{S_1 = \neg G, \Gamma_1 \rightarrow \Delta}\,(\neg \rightarrow)}{S = F, \neg G, \Gamma_1 \rightarrow \Delta}\,(w \rightarrow)$$

Now let's analyse this derivation:

$$\dfrac{\begin{array}{c}\mathcal{D}_2\\ S_2 = \Gamma_1 \rightarrow \Delta, G\end{array}}{S_3 = F, \Gamma_1 \rightarrow \Delta, G}\,(w \rightarrow)$$

The height of such derivation is less than h, therefore according to the induction hypothesis $(w \rightarrow)$ can be eliminated from the derivation to get the derivation $\mathcal{D}_3$. Finally the derivation of S without the application of $(w \rightarrow)$ is:

$$\dfrac{\begin{array}{c}\mathcal{D}_3\\ S_3 = F, \Gamma_1 \rightarrow \Delta, G\end{array}}{S = F, \neg G, \Gamma_1 \rightarrow \Delta}\,(\neg \rightarrow)$$

If the rule $(\rightarrow \Box_l)$ is applied to S_1 in $\mathcal{D}$, then if F is of the form $\Box_l H$, then this case is dealt with in the same way as the case of the logical rules. If F is not of the form $\Box_l H$, then $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_2\\ S_2 = \Gamma_2 \rightarrow G\end{array}}{S_1 = \Gamma_1, \Box_l \Gamma_2 \rightarrow \Delta_1, \Box_l G}\,(\rightarrow \Box_l)}{S = F, \Gamma_1, \Box_l \Gamma_2 \rightarrow \Delta_1, \Box_l G}\,(w \rightarrow)$$

In this case the application of $(w \to)$ rule can be eliminated even easier by constructing the following derivation of S:

$$\dfrac{\begin{matrix}\mathcal{D}_2 \\ S_2 = \Gamma_2 \to G\end{matrix}}{S = F, \Gamma_1, \Box_l \Gamma_2 \to \Delta_1, \Box_l G}\ (\to \Box_l)$$

This completes the proof of admissibility of the rule $(w \to)$. $\Box$

As described earlier, the derivation search in Gentzen-type calculi starts with the sequent, derivability of which is to be checked, and rules are backward applied until axiom(s) or sequent(s) such that no rules are backward applicable to them are obtained. The process could be described in a more formal way:

Definition 1.4.9. *A derivation search in sequent calculi consists of the following steps:*

1. *Let initial sequent be the root of the derivation search tree.*
2. *Let's take some leaf of the derivation search tree, which has not been analysed yet. Let's denote it S.*
3. *If S is an axiom and all the leafs have already been analysed, then the derivation search is completed. If all of the leafs are axioms, then S is derivable, otherwise it is not derivable.*
4. *If S is an axiom and there is at least one leaf, which is not yet analysed, then go to 2.*
5. *If S is not an axiom and at least one rule can be applied to it, then let's apply it. If there is more than one possibility, then let's choose one of them. After the application let all the premises be the child nodes of S. Go to 2.*
6. *If S is not an axiom and no rule can be applied to it, then let's analyse the branch of S. Let's go back the branch and let's find the first sequent such that several rules can be applied to it and at least one such application has not already been analysed. If such sequent exists, then let's denote it S'. Let's delete the tree above S'. Let's choose any application which has not been analysed and let the premises of such application be the new child nodes of S'. After that go to 2.*

7. *If S is not an axiom, no rule can be applied to it and the sequent described in Step 6 does not exist in the branch of S, then the derivation search is completed and S is not derivable.*

The part of the process described in Step 6 is called *backtracking* and is needed because as defined earlier sequent is derivable if at least one derivation search tree is a derivation tree. Now let's introduce one property of the rule.

Definition 1.4.10. *A rule of sequent calculus is invertible, iff from the fact that the conclusion of the rule is derivable, follows that all the premises of the rule are derivable.*

Invertibility is a very important property for backtracking. If in Step 5 there is a choice between several applications of invertible rules, then after applying one of them, there is no need to backtrack and check the other ones.

Lemma 1.4.11. *Logical rules are invertible in all the defined Gentzen-type calculi.*

Proof. Let's analyse only rule $(\neg \rightarrow)$ and only calculus GK_n. Other cases are analogous.

Let's say that sequent $S = \neg F, \Gamma \rightarrow \Delta$ is derivable in GK_n and the derivation tree of S is $\mathcal{D}$. It must be shown that sequent $S_1 = \Gamma \rightarrow \Delta, F$ is derivable in GK_n too. The proof is by induction on $\mathrm{h}(\mathcal{D})$.

If $\mathrm{h}(\mathcal{D}) = 1$, then S is an axiom. If $\neg F$ is not the main formula of S, then S is of the form $\neg F, G, \Gamma_1 \rightarrow \Delta_1, G$ and S_1 is of the form $G, \Gamma_1 \rightarrow \Delta_1, G, F$. It is obvious that S_1 is axiom of GK_n. Otherwise, if $\neg F$ is the main formula of S, then S is of the form $\neg F, \Gamma \rightarrow \Delta_1, \neg F$ and S_1 is of the form $\Gamma \rightarrow \Delta_1, \neg F, F$. Then the derivation tree of S_1 is obviously:

$$\frac{F, \Gamma \rightarrow \Delta_1, F}{\Gamma \rightarrow \Delta_1, \neg F, F}\ (\rightarrow \neg)$$

Suppose that lemma is valid, when $\mathrm{h}(\mathcal{D}) < h$. Let $\mathrm{h}(\mathcal{D}) = h$. Let's analyse all the possible bottom-most inferences of $\mathcal{D}$. If the main formula of the bottom-most inference is $\neg F$ and the rule $(\neg \rightarrow)$ is applied, then the derivation tree of S_1 is the same as $\mathcal{D}$ without the last inference.

Otherwise, if $(\neg \rightarrow)$ is applied, then $\mathcal{D}$ is:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \neg F, \Gamma_1 \rightarrow \Delta, G\end{array}}{S = \neg F, \neg G, \Gamma_1 \rightarrow \Delta}\ (\neg \rightarrow)$$

$\mathrm{h}(\mathcal{D}_1) < \mathrm{h}(\mathcal{D})$, therefore by induction hypothesis $\vdash_{GK_n} S_1^{'}$, where $S_1^{'} = \Gamma_1 \rightarrow \Delta, G, F$. Let the derivation tree of $S_1^{'}$ be $\mathcal{D}_2$. Then the derivation tree of S_1 is:

$$\dfrac{\begin{array}{c}\mathcal{D}_2\\ S_1^{'} = \Gamma_1 \rightarrow \Delta, G, F\end{array}}{S_1 = \neg G, \Gamma_1 \rightarrow \Delta, F}\ (\neg \rightarrow)$$

Other cases of logical rules are completely analogous.

If $(\rightarrow \Box_l)$ is applied, then $\mathcal{D}$ is:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_2 \rightarrow G\end{array}}{S = \neg F, \Gamma_1, \Box_l\Gamma_2 \rightarrow \Delta_1, \Box_l G}\ (\rightarrow \Box_l)$$

In this case the derivation tree of S_1 is:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_2 \rightarrow G\end{array}}{S_1 = \Gamma_1, \Box_l\Gamma_2 \rightarrow \Delta_1, \Box_l G, F}\ (\rightarrow \Box_l)$$

□

Lemma 1.4.12. *Rule* $(\Box \rightarrow)$ *is invertible in reflexive modal calculi GT and GS4. Rule* $(\Box_l \rightarrow)$ *is invertible in reflexive multimodal calculi* GT_n *and* $GS4_n$.

Proof. This lemma is direct corollary of Lemma 1.4.8. Let's analyse only rule $(\Box_l \rightarrow)$. The case of $(\Box \rightarrow)$ is completely analogous.

If sequent $\Box_l F, \Gamma \rightarrow \Delta$ is derivable and the derivation tree is $\mathcal{D}$, then the derivation tree of sequent $F, \Box_l F, \Gamma \rightarrow \Delta$ is:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Box_l F, \Gamma \rightarrow \Delta\end{array}}{F, \Box_l F, \Gamma \rightarrow \Delta}\ (w \rightarrow)$$

□

Now admissibility of two other rules can be proved.

Lemma 1.4.13. *The structural rules of contraction:*

$$\frac{F, F, \Gamma \to \Delta}{F, \Gamma \to \Delta}\,(c \to) \qquad \frac{\Gamma \to \Delta, F, F}{\Gamma \to \Delta, F}\,(\to c)$$

are admissible in all the defined Gentzen-type calculi. The main formula of these rules is F.

The notation of weakening and contraction structural rules as $(w \to)$, $(\to w)$, $(c \to)$ and $(\to c)$ is due to [51].

Proof. Once again let's analyse only calculus GK_n. The cases of other calculi are completely analogous. However, this time both rules must be analysed together. It is enough to show, that it is possible to eliminate the applications of contraction structural rules from the derivation trees.

Similar to the proof of Lemma 1.4.8, let's analyse only the derivation trees, which have only one application of the contraction structural rule, which is the bottom-most inference in the derivation tree. Afterwards the reasoning can be extended to every derivation tree by induction on the number of applications of contraction structural rules.

Let $\mathcal{D}$ be such derivation, sequent S be the conclusion of bottom-most inference, S_1 be the presumption and F be the main formula. The proof is by double induction on ordered pair $\langle \mathrm{l}(F), \mathrm{h}(\mathcal{D})\rangle$.

The induction base. If $\mathrm{h}(\mathcal{D}) = 2$, then S_1 is an axiom. If rule $(c \to)$ is applied and F is the main formula of the axiom, then S_1 is of the form $F, F, \Gamma \to \Delta, F$ and S is of the form $F, \Gamma \to \Delta, F$. It is obvious, that S is an axiom of GK_n and the application of $(c \to)$ is not needed. If rule $(c \to)$ is applied and F is not the main formula of the axiom, then S_1 is of the form $F, F, G, \Gamma \to \Delta, G$, sequent S is of the form $F, G, \Gamma \to \Delta, G$ and once again it is obvious, that S is an axiom of GK_n. The induction base of rule $(\to c)$ is analogous.

The induction step. Let $\mathrm{h}(\mathcal{D}) > 2$. First of all, let the bottom-most inference in $\mathcal{D}$ be an application of $(c \to)$ rule. Now let's check what rule is applied to S_1. All the cases of logical rules are similar, so let's analyse only $(\neg \to)$. If F is the main formula of such application, then $\mathcal{D}$ is:

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}_2\\ S_2 = \neg G, \Gamma \to \Delta, G\end{matrix}}{S_1 = \neg G, \neg G, \Gamma \to \Delta}\,(\neg \to)}{S = \neg G, \Gamma \to \Delta}\,(c \to)$$

Rule $(\neg\rightarrow)$ is invertible and $\vdash_{GK_n} S_2$, therefore sequent $S_3 = \Gamma \rightarrow \Delta, G, G$ is also derivable. Let the derivation tree of S_3 be $\mathcal{D}_3$. Now, let's check the following derivation:

$$\dfrac{\begin{array}{c}\mathcal{D}_3\\ S_3 = \Gamma \rightarrow \Delta, G, G\end{array}}{S_4 = \Gamma \rightarrow \Delta, G}\,(\rightarrow c)$$

According to induction hypothesis, because $\mathrm{l}(G) < \mathrm{l}(F)$, the application of the contraction structural rule can be eliminated from this derivation to get derivation $\mathcal{D}_3^{'}$. Now a derivation tree of S without contraction is:

$$\dfrac{\begin{array}{c}\mathcal{D}_3^{'}\\ S_4 = \Gamma \rightarrow \Delta, G\end{array}}{S = \neg G, \Gamma \rightarrow \Delta}\,(\neg\rightarrow)$$

If F is not the main formula of $(\neg\rightarrow)$, then $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_2\\ S_2 = F, F, \Gamma \rightarrow \Delta, G\end{array}}{S_1 = F, F, \neg G, \Gamma \rightarrow \Delta}\,(\neg\rightarrow)}{S = F, \neg G, \Gamma \rightarrow \Delta}\,(c\rightarrow)$$

Now let's analyse this derivation:

$$\dfrac{\begin{array}{c}\mathcal{D}_2\\ S_2 = F, F, \Gamma \rightarrow \Delta, G\end{array}}{S_3 = F, \Gamma \rightarrow \Delta, G}\,(c\rightarrow)$$

The height of such derivation is less than the height of $\mathcal{D}$ and the main formula of contraction is the same, therefore according to the induction hypothesis $(c\rightarrow)$ can be eliminated from the derivation to get the derivation $\mathcal{D}_2^{'}$. Finally the derivation tree of S without contraction is:

$$\dfrac{\begin{array}{c}\mathcal{D}_2^{'}\\ S_3 = F, \Gamma \rightarrow \Delta, G\end{array}}{S = F, \neg G, \Gamma \rightarrow \Delta}\,(\neg\rightarrow)$$

If the rule $(\rightarrow \Box_l)$ is applied to S_1 in $\mathcal{D}$, then if F is of the form $\Box_l H$, then this case is dealt with in the same way as the case of the logical rules. If F is not of the form $\Box_l H$, then $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_2\\ S_2 = \Gamma_2 \rightarrow G\end{array}}{S_1 = F, F, \Gamma_1, \Box_l\Gamma_2 \rightarrow \Delta, \Box_l G}\,(\rightarrow \Box_l)}{S = F, \Gamma_1, \Box_l\Gamma_2 \rightarrow \Delta, \Box_l G}\,(c\rightarrow)$$

In this case the application of $(c \rightarrow)$ rule can be eliminated even easier by constructing the following derivation tree of S:

$$\dfrac{\begin{matrix}\mathcal{D}_2\\ S_2 = \Gamma_2 \rightarrow G\end{matrix}}{S = F, \Gamma_1, \Box_l\Gamma_2 \rightarrow \Delta, \Box_l G}\ (\rightarrow \Box_l)$$

Now let the bottom-most inference in $\mathcal{D}$ be an application of $(\rightarrow c)$ rule. Once again, let's check what rule is applied to S_1. All the cases of logical rules are analogous to the $(c \rightarrow)$ part of the proof. Thus let rule $(\rightarrow \Box_l)$ be applied to S_1. If F is the main formula of application of $(\rightarrow \Box_l)$, then $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_2\\ S_2 = \Gamma_2 \rightarrow G\end{matrix}}{S_1 = \Gamma_1, \Box_l\Gamma_2 \rightarrow \Delta, \Box_l G, \Box_l G}\ (\rightarrow \Box_l)}{S = \Gamma_1, \Box_l\Gamma_2 \rightarrow \Delta, \Box_l G}\ (\rightarrow c)$$

To eliminate the application of $(\rightarrow c)$ rule, let's construct the derivation:

$$\dfrac{\begin{matrix}\mathcal{D}_2\\ S_2 = \Gamma_2 \rightarrow G\end{matrix}}{S = \Gamma_1, \Box_l\Gamma_2 \rightarrow \Delta, \Box_l G}\ (\rightarrow \Box_l)$$

If F is not the main formula of application of $(\rightarrow \Box_l)$, then $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_2\\ S_2 = \Gamma_2 \rightarrow G\end{matrix}}{S_1 = \Gamma_1, \Box_l\Gamma_2 \rightarrow \Delta, \Box_l G, F, F}\ (\rightarrow \Box_l)}{S = \Gamma_1, \Box_l\Gamma_2 \rightarrow \Delta, \Box_l G, F}\ (\rightarrow c)$$

Similarly, to eliminate the application of $(\rightarrow c)$ rule, let's construct the derivation:

$$\dfrac{\begin{matrix}\mathcal{D}_2\\ S_2 = \Gamma_2 \rightarrow G\end{matrix}}{S = \Gamma_1, \Box_l\Gamma_2 \rightarrow \Delta, \Box_l G, F}\ (\rightarrow \Box_l)$$

This completes the proof of admissibility of contraction structural rules.

$\Box$

As it is proved in Lemma 1.4.11 all the rules of *GPC* are invertible. Therefore, no backtracking is needed in that calculus at all. However all of the modal and multimodal sequent calculi have at least one rule, which is not invertible. In all the modal calculi this rule is $(\rightarrow \Box)$ and in all the multimodal calculi it is $(\rightarrow \Box_l)$.

But there is a more important problem in modal and multimodal sequent calculi than backtracking. In some cases the derivation search does not terminate and may form infinite derivation search trees. The calculi, where this is not possible are called *terminating* (this term is used in e.g. [10]). More formally:

Definition 1.4.14. *A Gentzen-type calculus is terminating if every sequent has finite number of derivation search trees and it is not possible to construct an infinite derivation search tree of any sequent.*

It is said that derivation search tree is *complete*, if any branch ends with an axiom or a sequent, for which no rule of the calculus could be applied. It is not hard to see that every derivation tree is complete.

A terminating calculus provides an easy way to say if the sequent is derivable, or not. The procedure is simply to check all the possible complete derivation search trees of the sequent. If one of them is a derivation tree, then it is derivable, otherwise it is not derivable. Such check can be done in the same way as described in Definition 1.4.9, however in the case of terminating calculus the process is guaranteed to terminate.

It is easy to show (for example, using technique provided in Theorem 3.1.3), that calculi *GPC*, *GK* and GK_n are terminating. The finite derivation search in calculi GT and GT_n can be achieved by minor modification of the calculus (see calculus $G^{*}T^{c}_{n}$ defined in Definition 3.2.1 and Theorem 3.2.12). However to obtain terminating sequent calculi for transitive logics such as *GK4*, *GS4*, $GK4_n$ and $GS4_n$ more elaborate methods are needed.

Chapter 2

Basic Calculi for Multimodal Logics with Interaction

Interaction between agents is modelled in different ways. In this chapter the system with central processing unit, described in [38], is analysed. Speaking in terms of multimodal logic, one of the agents is called the central agent and it knows everything that is known to other agents. Additionally, only the systems, consisting of three or more agents (one central agent and at least two other agents) are analysed, because otherwise the situation can be modelled by monomodal logic.

2.1 Central Agent Axiom

In this dissertation central agent is denoted by letter c. Other agents are numbered as usual. Letter a is used to denote any agent, except the central one and letter l to mean any agent.

Definition 2.1.1. *The central agent axiom is:*

(C). $\Box_a F \supset \Box_c F$;

Of course, the central agent axiom has also some restrictions to the Kripke structure.

Lemma 2.1.2. *The central agent axiom is valid in every Kripke structure with frame* $\langle \mathcal{W}, \mathcal{R}_c, \mathcal{R}_1, \ldots, \mathcal{R}_n \rangle$ *iff* $\mathcal{R}_c \subseteq \bigcap_{a\in[1,n]} \mathcal{R}_a$.

Proof. Let's say that the central agent axiom is valid in every Kripke structure with frame $\langle \mathcal{W}, \mathcal{R}_c, \mathcal{R}_1, \ldots, \mathcal{R}_n \rangle$, but $\mathcal{R}_c \not\subseteq \bigcap_{a\in[1,n]} \mathcal{R}_a$. Therefore there are two worlds $w_1, w_2 \in \mathcal{W}$ such that $(w_1, w_2) \in \mathcal{R}_c$, but $(w_1, w_2) \notin \mathcal{R}_{a'}$

for some $a^{'} \in [1, n]$. Now lets define interpretation Φ in such way that for some propositional variable p $\Phi(w_2, p) = \bot$ and $\Phi(w, p) = \top$ for each $w \in \mathcal{W}, w \neq w_2$. Let's inspect Kripke structure $(S) = \langle \mathcal{W}, \mathcal{R}_c, \mathcal{R}_1, \dots, \mathcal{R}_n, \Phi \rangle$. Because $\Phi(w_2, p) = \bot$ and $(w_1, w_2) \in \mathcal{R}_c$, it is obvious that $\mathcal{S}, w_1 \nvDash \Box_c p$. However, $(w_1, w_2) \notin \mathcal{R}_{a'}$ and p is true in every world, except w_2, therefore $\mathcal{S}, w_1 \vDash \Box_{a'} p$. Thus $\mathcal{S}, w_1 \nvDash \Box_{a'} p \supset \Box_c p$ and this contradicts the presumption that the central agent axiom is valid in every Kripke structure with frame $\langle \mathcal{W}, \mathcal{R}_c, \mathcal{R}_1, \dots, \mathcal{R}_n \rangle$.

Now, let's say that $\mathcal{S} = \langle \mathcal{W}, \mathcal{R}_c, \mathcal{R}_1, \dots, \mathcal{R}_n, \Phi \rangle$ is a Kripke structure and $\mathcal{R}_c \subseteq \bigcap_{a \in [1,n]} \mathcal{R}_a$. Suppose, that for some world $w \in \mathcal{W}$ and some agent $a^{'} \in [1, n]$ it is true that $\mathcal{S}, w \vDash \Box_{a'} F$. According to the Definition 1.3.3, $\mathcal{S}, w_1 \vDash F$ for each w_1 such that $(w, w_1) \in \mathcal{R}_{a'}$. Now according to presumption for every $(w, w_2) \in \mathcal{R}_c$ it is true that $(w, w_2) \in \bigcap_{a \in [1,n]} \mathcal{R}_a$ and therefore $(w, w_2) \in \mathcal{R}_{a'}$. Therefore, $\mathcal{S}, w_2 \vDash F$ for each w_2 such that $(w, w_2) \in \mathcal{R}_c$. Hence according to the Definition 1.3.3 $\mathcal{S}, w \vDash \Box_c F$ and $\mathcal{S}, w \vDash \Box_{a'} F \supset \Box_c F$. Contrary, if $\mathcal{S}, w \nvDash \Box_{a'} F$, then it is also true that $\mathcal{S}, w \vDash \Box_{a'} F \supset \Box_c F$. Therefore, $\Box_{a'} F \supset \Box_c F$ is true in every world of $\mathcal{S}$ for any agent $a^{'}$. □

However it is possible to formulate even stricter requirement for this axiom, but before that let's prove another lemma.

Lemma 2.1.3. *Some formula is valid in every Kripke structure with frame $\langle \mathcal{W}, \mathcal{R}_c, \mathcal{R}_1, \dots, \mathcal{R}_n \rangle$ such that $\mathcal{R}_c \subseteq \bigcap_{a \in [1,n]} \mathcal{R}_a$ iff it is valid in every Kripke structure with frame $\left\langle \mathcal{W}^{'}, \mathcal{R}_c^{'}, \mathcal{R}_1^{'}, \dots, \mathcal{R}_n^{'} \right\rangle$ such that $\mathcal{R}_c^{'} \equiv \bigcap_{a \in [1,n]} \mathcal{R}_a^{'}$.*

Proof. The idea of this proof is sketched in [11] and is presented in full in [12].

If some formula is valid in every Kripke structure with requirement $\mathcal{R}_c \subseteq \bigcap_{a \in [1,n]} \mathcal{R}_a$, then obviously it is valid in every Kripke structure with stricter requirement $\mathcal{R}_c^{'} \equiv \bigcap_{a \in [1,n]} \mathcal{R}_a^{'}$.

To prove the opposite, let's say that formula F is valid in every Kripke structure with frame $\langle \mathcal{W}^{\circ}, \mathcal{R}_c^{\circ}, \mathcal{R}_1^{\circ}, \dots, \mathcal{R}_n^{\circ} \rangle$ such that $\mathcal{R}_c^{\circ} \equiv \bigcap_{a \in [1,n]} \mathcal{R}_a^{\circ}$, however there is a Kripke structure $\mathcal{S} = \langle \mathcal{W}, \mathcal{R}_c, \mathcal{R}_1, \dots, \mathcal{R}_n, \Phi \rangle$ such that $\mathcal{R}_c \subset \bigcap_{a \in [1,n]} \mathcal{R}_a$ and $\mathcal{S} \nvDash F$.

Now, let's form another Kripke structure. Let $\mathcal{W}_1^{'} = \mathcal{W}$ and $\mathcal{W}_{k+1}^{'}$ be a set of states $w_{w_1, w_2, l}$ for each $w_1 \in \mathcal{W}$, each $w_2 \in \mathcal{W}_k^{'}$ and each agent $l \in \{c\} \cup [1, n]$. Let $\mathcal{W}^{'} = \bigcup_i \mathcal{W}_i^{'}$. Let's define function $f : \mathcal{W}^{'} \to \mathcal{W}$ as follows: $f(w) = w$, if $w \in \mathcal{W}_1^{'}$ and $f(w_{w_1, w_2, l}) = w_1$, if $w_{w_1, w_2, l} \in \mathcal{W}_j^{'}, j \geqslant 2$.

Next, let $\mathcal{R}_l^{''} = \{(w_1, w_{w,w_1,l}) : (f(w_1), f(w_{w,w_1,l})) \in \mathcal{R}_l\}$ for each agent $l \in \{c\} \cup [1,n]$. Let $\mathcal{R}_a^{'} = \mathcal{R}_a^{''} \cup \mathcal{R}_c^{''}$ for each $a \in [1,n]$ and $\mathcal{R}_c^{'} = \mathcal{R}_c^{''}$. Let's notice, that this definition ensures that $\mathcal{R}_c^{'} \equiv \bigcap_{a\in[1,n]} \mathcal{R}_a^{'}$.

Finally, let $\Phi^{'}(w,p) = \Phi(f(w),p)$ for each $w \in \mathcal{W}^{'}$ and each propositional variable p of F.

Now let's analyse structure $\mathcal{S}^{'} = \left\langle \mathcal{W}^{'}, \mathcal{R}_c^{'}, \mathcal{R}_1^{'}, \ldots, \mathcal{R}_n^{'}, \Phi^{'} \right\rangle$. The aim is to show, that $\mathcal{S}^{'} \nvDash F$. However it can be proved that for every world $w \in \mathcal{W}^{'}$ it is true that $\mathcal{S}^{'}, w \vDash G$ iff $\mathcal{S}, f(w) \vDash G$. The proof is by induction on the form of G. If G is a propositional variable, then this is obvious, because of the way $\Phi^{'}$ is defined.

If G is of the form $G_1 \wedge G_2$, then by induction hypothesis $\mathcal{S}^{'}, w \vDash G_1$ iff $\mathcal{S}, f(w) \vDash G_1$ and $\mathcal{S}^{'}, w \vDash G_2$ iff $\mathcal{S}, f(w) \vDash G_2$. Therefore, if $\mathcal{S}, f(w) \vDash G$, then both $\mathcal{S}, f(w) \vDash G_1$ and $\mathcal{S}, f(w) \vDash G_2$ and therefore $\mathcal{S}^{'}, w \vDash G_1$ and $\mathcal{S}^{'}, w \vDash G_2$. From this follows that $\mathcal{S}^{'}, w \vDash G$. Otherwise, if $\mathcal{S}, f(w) \nvDash G$, then $\mathcal{S}, f(w) \nvDash G_1$ or $\mathcal{S}, f(w) \nvDash G_2$ and therefore $\mathcal{S}^{'}, w \nvDash G_1$ or $\mathcal{S}^{'}, w \nvDash G_2$. From this follows that $\mathcal{S}^{'}, w \nvDash G$. The cases, when G is of the form $\neg G_1$, $G_1 \vee G_2$ and $G_1 \supset G_2$ are analogous.

If G is of the form $\Box_{a'} G_1, a^{'} \in [1,n]$, then by induction hypothesis for each $w_1 \in \mathcal{W}^{'}$ it is true that $\mathcal{S}^{'}, w_1 \vDash G_1$ iff $\mathcal{S}, f(w_1) \vDash G_1$. Now if $\mathcal{S}, f(w) \vDash G$, then for each $(f(w), w_2) \in \mathcal{R}_{a'}$ it is true that $\mathcal{S}, w_2 \vDash G_1$. By the construction of $\mathcal{R}_{a'}^{'}$, if $(w, w_1) \in \mathcal{R}_{a'}^{'}$, then $(f(w), f(w_1))$ is either part of $\mathcal{R}_{a'}$ or $\mathcal{R}_c$. Now because of the presumption made in the beginning of the proof, $\mathcal{R}_c \subset \bigcap_{a\in[1,n]} \mathcal{R}_a$ and therefore definitely $(f(w), f(w_1)) \in \mathcal{R}_{a'}$. From this it follows that $\mathcal{S}^{'}, w_1 \vDash G_1$ for every $(w, w_1) \in \mathcal{R}_{a'}^{'}$ and therefore $\mathcal{S}^{'}, w \vDash G$.

Otherwise, if $\mathcal{S}, f(w) \nvDash G$, then there is a world w_2 such that $(f(w), w_2) \in \mathcal{R}_{a'}$, but $\mathcal{S}, w_2 \nvDash G_1$. Due to the way $\mathcal{R}_{a'}^{'}$ is constructed and because $f(w_{w_2,w,a'}) = w_2$, it is true that $(w, w_{w_2,w,a'}) \in \mathcal{R}_{a'}^{'}$. According to the induction hypothesis, $\mathcal{S}^{'}, w_{w_2,w,a'} \nvDash G_1$ and therefore $\mathcal{S}^{'}, w \nvDash G$.

The case, where G is of the form $\Box_c G_1$ is analogous to the previous one.

The direct consequence of this proof and the assumption that $\mathcal{S} \nvDash F$ is that $\mathcal{S}^{'} \nvDash F$. However, recall that $\mathcal{R}_c^{'} \equiv \bigcap_{a\in[1,n]} \mathcal{R}_a^{'}$ and therefore this is a contradiction of the assumption that F is valid in every Kripke structure with such frame. $\Box$

The property $\mathcal{R}_c^{'} \equiv \bigcap_{a\in[1,n]} \mathcal{R}_a^{'}$ is called the central agent property. Note that the proof of Lemma 2.1.2 does not take into account any additional requirements that are added together with axioms *(T_l)*, *(4_l)*. Moreover, it is not hard to modify the proof of Lemma 2.1.3 to deal with reflexive, transitive, reflexive and transitive logics. Therefore, central agent property is independent of these requirements.

2.2 Hilbert-type Calculi

Definition 2.2.1. *Hilbert-type calculi for multimodal logics K_n, T_n, $K4_n$ and $S4_n$ with central agent axiom (respectively K_n^c, T_n^c, $K4_n^c$ and $S4_n^c$) are defined respectively: $HK_n^c = HK_n + (\mathrm{C})$, $HT_n^c = HT_n + (\mathrm{C})$, $HK4_n^c = HK4_n +$ (C) and $HS4_n^c = HS4_n + (\mathrm{C})$.*

As mentioned above, the central agent axiom incorporates one more restriction in to the definition of validity of the formula: central agent property. Therefore:

- $\vDash_{K_n^c} F$ iff for any Kripke structure $\mathcal{S}$ with frame that satisfies central agent property $\mathcal{S} \vDash F$.
- $\vDash_{T_n^c} F$ iff for any Kripke structure $\mathcal{S}$ with reflexive frame that satisfies central agent property $\mathcal{S} \vDash F$.
- $\vDash_{K4_n^c} F$ iff for any Kripke structure $\mathcal{S}$ with transitive frame that satisfies central agent property $\mathcal{S} \vDash F$.
- $\vDash_{S4_n^c} F$ iff for any Kripke structure $\mathcal{S}$ with reflexive and transitive frame that satisfies central agent property $\mathcal{S} \vDash F$.

To show the soundness and completeness of the defined calculi, the concept of distributed knowledge is used as it is defined in [12]. Firstly, the distributed knowledge operator D is incorporated in the language in a similar way as knowledge operator $\Box_l$ in Definition 1.3.1. Here formula DF means that all the agents have distributed knowledge of F. Now for Kripke structure $\mathcal{S} = \langle \mathcal{W}, \mathcal{R}_1, \ldots, \mathcal{R}_n, \Phi \rangle$ and world $w \in \mathcal{W}$ the truth of formula DF is defined as follows: $\mathcal{S}, w \vDash DF$ iff $\mathcal{S}, w_1 \vDash F$ for all $w_1 \in \mathcal{W}$ such that $(w, w_1) \in \bigcap_{a \in [1,n]} \mathcal{R}_a$.

To enrich the Hilbert type calculus with the operator of distributed knowledge one axiom is used: $\Box_a F \supset DF$. What is more, if there is only one agent ($n = 1$), then axiom $DF \supset \Box_1 F$ is also included. Finally, the distributed knowledge operator must satisfy the axiom *(K_l)* and other axioms, that are part of the logic[1]. Now it is not hard to see that central agent knowledge operator $\Box_c$ satisfy all the conditions set out for distributed logic operator, if there are more than one agent except the central one.

However in [12] only modal logic $S5_n$ is analysed. Wider selection of knowledges is discussed in [11], but there the definition of distributed knowledge is a little bit different. In [11] distributed knowledge of some subset of agents is also analysed. Nevertheless, it is not hard to argue that both definitions are equivalent as far as distributed knowledge of all the agents is concerned.

What is more, the proofs of soundness and completeness of Hilbert-type calculi for different multimodal logics enriched with distributed knowledge operator are provided in [11, 12]. From this and from the discussion above it can be concluded that:

Corollary 2.2.2. *Calculi HK^c_n, $HK4^c_n$, HT^c_n and $HS4^c_n$ are sound and complete.*

2.3 Gentzen-type Calculi with Cut

Definition 2.3.1. *Gentzen-type calculi with cut for multimodal logics with central agent axiom K^c_n, $K4^c_n$, T^c_n and $S4^c_n$ (respectively, GK^c_ncut, $GK4^c_n$cut, GT^c_ncut and $GS4^c_n$cut) are obtained from respective multimodal Gentzen-type calculi GK_n, $GK4_n$, GT_n and $GS4_n$ by adding the cut rule:*

$$\frac{\Gamma \to \Delta, F \qquad F, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}\ (cut\ F)$$

and modal rule for central agent operator $\Box_c$, which depends on the calculus.

In the case of GK^c_ncut and GT^c_ncut:

$$\frac{\Gamma_2 \to F}{\Gamma_1, \Box_* \Gamma_2 \to \Delta, \Box_c F}\ (\to \Box^c)$$

In the case of $GK4^c_n$cut:

$$\frac{\Gamma_2, \Box_* \Gamma \to F}{\Gamma_1, \Box_* \Gamma_2 \to \Delta, \Box_c F}\ (\to \Box^c)$$

[1]For example distributed knowledge operator in logic $S4_n$ additionally satisfy axioms *(T_l)* and *(4_l)*. Therefore, $D(F \supset G) \supset (DF \supset DG)$, $DF \supset F$ and $DF \supset DDF$ are axioms of the calculus.

In the case of $GS4_n$:

$$\frac{\Box_* \Gamma_2 \to F}{\Gamma_1, \Box_* \Gamma_2 \to \Delta, \Box_c F} \ (\to \Box^c)$$

Here $\Box_* \Gamma_2 = \Box_{l_1} F_1, \ldots, \Box_{l_k} F_k$, where $k \geqslant 0$ and l_j is any agent for every $j \in [1, k]$.

In the (cut F) rule F is called the *cut formula*. In the $(\to \Box^c)$ rules, $\Box_c F$ is main formula and F is side formula.

In a similar way as in Lemma 1.4.8 it is possible to prove the following.

Lemma 2.3.2. *The structural rules of weakening are admissible in $GK^c_n cut$, $GK4^c_n cut$, $GT^c_n cut$ and $GS4^c_n cut$.*

It is easy to prove that all the rules except (cut F), $(\to \Box_l)$ and $(\to \Box^c)$ are invertible in $GK^c_n cut$, $GK4^c_n cut$, $GT^c_n cut$ and $GS4^c_n cut$. The following lemma formally proves only the invertibility of $(\to\supset)$ and the proof for other rules is analogous.

Lemma 2.3.3. *The rule $(\to\supset)$ is invertible in $GK^c_n cut$, $GK4^c_n cut$, $GT^c_n cut$ and $GS4^c_n cut$. That is, $\vdash_{\mathcal{C}} \Gamma \to \Delta, F \supset G$, iff $\vdash_{\mathcal{C}} F, \Gamma \to \Delta, G$, where $\mathcal{C} \in \{GK^c_n cut, GK4^c_n cut, GT^c_n cut, GS4^c_n cut\}$.*

Proof. All the mentioned calculi are very similar, therefore, let's analyse all of them together in one proof. For clarity, the parts of the proof, where the difference between them matters, are mentioned separately.

If $\vdash_{\mathcal{C}} F, \Gamma \to \Delta, G$ and the derivation tree of this sequent is $\mathcal{D}$, then the derivation tree of $\vdash_{\mathcal{C}} \Gamma \to \Delta, F \supset G$ is:

$$\frac{\begin{array}{c}\mathcal{D}\\ F, \Gamma \to \Delta, G\end{array}}{\Gamma \to \Delta, F \supset G} \ (\to\supset)$$

To prove the other part, let's apply induction on the height of the derivation tree $\mathcal{D}$ of $S = \Gamma \to \Delta, F \supset G$.

If $\mathrm{h}(\mathcal{D}) = 1$, then S is an axiom. If $F \supset G$ is not the main formula of the axiom, then S is of the form $H, \Gamma_1 \to \Delta_1, H, F \supset G$, and it is obvious that $S_1 = F, \Gamma \to \Delta, G$ is also an axiom, because it is equal to $F, H, \Gamma_1 \to \Delta_1, H, G$. If $F \supset G$ is the main formula of the axiom, then S is of the form $F \supset G, \Gamma_1 \to \Delta, F \supset G$ and S_1 is of the form

$F, F \supset G, \Gamma_1 \to \Delta, G$. However, this sequent is also derivable and the derivation tree is:

$$\dfrac{F, \Gamma_1 \to \Delta, G, F \qquad F, G, \Gamma_1 \to \Delta, G}{F, F \supset G, \Gamma_1 \to \Delta, G} \; (\supset\to)$$

Now let's say that the lemma holds if the height of the derivation tree is less than h. Let $\mathrm{h}(\mathcal{D}) = h$. Let's show, that $S_1 = F, \Gamma \to \Delta, G$ is also derivable. In order to do that it is enough to go through all the possible bottom-most inferences of $\mathcal{D}$.

If the bottom-most inference of $\mathcal{D}$ is an application of $(\to\supset)$ and $F \supset G$ is the main formula, then the premise of such application is sequent S_1 and therefore it is derivable. Otherwise, formula $F \supset G$ is not the main formula and there are three cases. Firstly, if the bottom-most application is of some logical rule or rule $(\Box_l \to)$, then the formula $F \supset G$ is part of the succedent of the premise of such application. As an example let's take the rule $(\neg \to)$. Thus $\mathcal{D}$ is of the form:

$$\dfrac{\begin{matrix}\mathcal{D}_2 \\ S_2 = \Gamma_1 \to \Delta, F \supset G, H\end{matrix}}{S = \neg H, \Gamma_1 \to \Delta, F \supset G} \; (\neg \to)$$

Because $\mathrm{h}(\mathcal{D}_2) < h$ according to the induction hypothesis, sequent $S_3 = F, \Gamma \to \Delta, G, H$ is also derivable. Let $\mathcal{D}_3$ denote the derivation tree of S_3. Now the derivation tree of S_1 is:

$$\dfrac{\begin{matrix}\mathcal{D}_3 \\ S_3 = F, \Gamma_1 \to \Delta, G, H\end{matrix}}{S_1 = \neg H, F, \Gamma_1 \to \Delta, G} \; (\neg \to)$$

Otherwise, if the bottom-most inference is application of the rule $(\to \Box_l)$ or $(\to \Box^c)$, then $F \supset G$ is definitely not part of the premise. Let's analyse only the $(\to \Box_l)$ of $GK_n^c cut$ case because the other cases are analogous. In that case, $\mathcal{D}$ is of the form:

$$\dfrac{\begin{matrix}\mathcal{D}_2 \\ S_2 = \Gamma_2 \to H\end{matrix}}{S = \Gamma_1, \Box_l \Gamma_2 \to \Delta_1, \Box_l H, F \supset G} \; (\to \Box_l)$$

Now, if F is not of the form $\Box_l F_1$, then the derivation tree of S_1 is:

$$\dfrac{\begin{matrix}\mathcal{D}_2 \\ S_2 = \Gamma_2 \to H\end{matrix}}{S_1 = F, \Gamma_1, \Box_l \Gamma_2 \to \Delta_1, \Box_l H, G} \; (\to \Box_l)$$

Otherwise, because of the admissibility of $(w \to)$, the derivation tree of S_1 is:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_2\\ S_2 = \Gamma_2 \to H\end{array}}{S_3 = F_1, \Gamma_2 \to H}\,(w \to)}{S_1 = F, \Gamma_1, \Box_l \Gamma_2 \to \Delta_1, \Box_l H, G}\,(\to \Box_l)$$

Finally, if the bottom-most inference is application of the rule (cut H), then either $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\mathcal{D}_2\\ S_2 = \Gamma_1 \to \Delta_1, H\end{array} \qquad \begin{array}{c}\mathcal{D}_3\\ S_3 = H, \Pi \to \Lambda, F \supset G\end{array}}{S = \Gamma_1, \Pi \to \Delta_1, \Lambda, F \supset G}\,(\text{cut } H)$$

or $F \supset G$ is part of S_2, but not of S_3. Let's analyse only the displayed case, because the other one is completely analogous. In this case, because $\mathrm{h}(\mathcal{D}_3) < h$ according to the induction hypothesis, $S_4 = H, F, \Pi \to \Lambda, G$ is also derivable. Let $\mathcal{D}_4$ denote the derivation tree of S_4. Now the derivation tree of S_1 is:

$$\dfrac{\begin{array}{c}\mathcal{D}_2\\ S_2 = \Gamma_1 \to \Delta_1, H\end{array} \qquad \begin{array}{c}\mathcal{D}_4\\ S_4 = H, F, \Pi \to \Lambda, G\end{array}}{S_1 = F, \Gamma_1, \Pi \to \Delta_1, \Lambda, G}\,(\text{cut } H)$$

$\Box$

Now the soundness and completeness of the sequent calculi can be proved. But instead of semantic discussion, this time the soundness and completeness of the respective Hilbert-type calculi is used. So the equivalence between the Hilbert-type and Gentzen-type calculi is shown. Let's start by proving the completeness.

Lemma 2.3.4. *If some formula is derivable in HK^c_n ($HK4^c_n$, HT^c_n, $HS4^c_n$), then it is derivable in $GK^c_n cut$ (respectively $GK4^c_n cut$, $GT^c_n cut$ or $GS4^c_n cut$).*

Proof. Once again, let's analyse only HK^c_n case. The other ones can be proven analogously.

Let formula F_n be derivable in HK^c_n and let $\mathcal{D} = \{F_1, \dots, F_n\}$ be the derivation of F_n. The proof, that $\vdash_{GK^c_n cut} F_n$ is by induction on the height of $\mathcal{D}$.

The induction base. If $\mathrm{h}(\mathcal{D}) = 1$, then F_n is an axiom of HK^c_n. This derivation is replaced by derivation of $\to F_n$ in $GK^c_n cut$. It is easy to show that all the axioms of HK^c_n are derivable in $GK^c_n cut$. Only one case (the central agent axiom) is demonstrated. Another case can be found in Example 1.1.11.

$$\cfrac{\cfrac{F \rightarrow F}{\Box_a F \rightarrow \Box_c F}\ (\rightarrow \Box^c)}{\rightarrow \Box_a F \supset \Box_c F}\ (\rightarrow\supset)$$

All the axioms of all the considered Hilbert-type calculi are derivable in respective Gentzen-type calculi.

The induction step. If all the derivations in HK_n^c with the height less than h can be replaced by the derivation trees in $GK_n^c cut$, then suppose that $\mathrm{h}(\mathcal{D}) = h$ and let's analyse the last application of the rule in $\mathcal{D}$. In fact, there are only two cases:

1. The modus ponens rule. In this case F_n is obtained from F_i and $F_j = F_i \supset F_n$, where $i, j < n$. Let $\mathcal{D}_i = \{F_1, \ldots, F_i\}$ and $\mathcal{D}_j = \{F_1, \ldots, F_j\}$. By the definition, $\mathrm{h}(\mathcal{D}_i) < h$ and $\mathrm{h}(\mathcal{D}_j) < h$, so according to the induction hypothesis, $\mathcal{D}_i$ and $\mathcal{D}_j$ can be transformed to the derivation trees of $\rightarrow F_i$ and $\rightarrow F_i \supset F_n$ in $GK_n^c cut$. Assume that after this transformation, the derivation trees $\mathcal{D}_i^{'}$ and $\mathcal{D}_j^{'}$ are obtained. According to Lemma 2.3.3, if $\vdash_{GK_n^c cut} \rightarrow F_i \supset F_n$, then also $\vdash_{GK_n^c cut} F_i \rightarrow F_n$, so let $\mathcal{D}_j^{''}$ be the derivation tree of the latter. Now the derivation tree of F_n in $GK_n^c cut$ is as follows:

$$\cfrac{\begin{array}{c}\mathcal{D}_i^{'}\\ \rightarrow F_i\end{array} \qquad \begin{array}{c}\mathcal{D}_j^{''}\\ F_i \rightarrow F_n\end{array}}{\rightarrow F_n}\ (\text{cut } F_i)$$

2. The rule of necessity. Then $F_n = \Box_l F_i$ is obtained from F_i, where $i < n$. Let $\mathcal{D}_i = \{F_1, \ldots, F_i\}$. Since $\mathrm{h}(\mathcal{D}_i) < k$, according to the induction hypothesis it can be replaced by derivation tree $\mathcal{D}_i^{'}$ of $\rightarrow F_i$ in $GK_n^c cut$ and the whole derivation can be replaced by:

$$\cfrac{\begin{array}{c}\mathcal{D}_i^{'}\\ \rightarrow F_i\end{array}}{\rightarrow \Box_l F_i}\ (\rightarrow \Box_l)$$

$\Box$

The direct corollary of this lemma is the completeness of the sequent calculi.

Theorem 2.3.5. *Calculi $GK_n^c cut$, $GK4_n^c cut$, $GT_n^c cut$ and $GS4_n^c cut$ are complete.*

Proof. Let S be some sequent. If $\vDash_{\mathcal{L}} S$, then $\vDash_{\mathcal{L}} \mathrm{Cor}(S)$. Thus by Lemma 2.3.4 and completeness of Hilbert-type calculi it is true that $\vdash_{\mathcal{C}} \rightarrow \mathrm{Cor}(S)$. Now, it is possible to apply the rules $(\wedge \rightarrow)$, $(\rightarrow \vee)$, $(\rightarrow\supset)$ and $(\rightarrow \neg)$ to sequent $\rightarrow \mathrm{Cor}(S)$ to get sequent S. Because the mentioned rules are invertible, it is true that $\vdash_{\mathcal{C}} S$. Here $\mathcal{L} = K_n^c$ ($K4_n^c$, T_n^c, $S4_n^c$) and $\mathcal{C} = GK_n^c cut$ (respectively $GK4_n^c cut$, $GT_n^c cut$ or $GS4_n^c cut$). $\square$

To show the soundness once again Hilbert-type calculi are used. However to shorten the proof, the derivability of some formulas are shown beforehand.

Lemma 2.3.6. *The formulas are divided according to the set of calculi, that can derive them.*

1. *The following formulas are derivable in HK_n^c, $HK4_n^c$, HT_n^c and $HS4_n^c$:*
   (a) $(F \vee G) \supset (G \vee F)$.
   (b) $(F \supset G) \supset (\neg F \vee G)$.
   (c) $(F \vee G) \supset (\neg F \supset G)$.
   (d) $(\neg F \wedge \neg G) \supset \neg(F \vee G)$.
   (e) $\neg(F \wedge G) \supset (\neg F \vee \neg G)$.
   (f) $(\Box_l F_1 \wedge \ldots \wedge \Box_l F_n) \supset \Box_l(F_1 \wedge \ldots \wedge F_n)$, *where* $n \geqslant 1$.
   (g) $(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_n} F_n) \supset (\Box_c F_1 \wedge \ldots \wedge \Box_c F_n)$, *where* $n \geqslant 1$.
2. *The following formula is derivable in $HK4_n^c$ and $HS4_n^c$:*
   (a) $(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_n} F_n) \supset (\Box_{l_1}\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_n}\Box_{l_n} F_n)$, *where* $n \geqslant 1$.

The complete proofs of derivability of those formulas are lengthy and therefore provided in the Appendix A.

Lemma 2.3.7. *The rules of commutativity of disjunction (*CD*), implication removal (*IR*) and implication introduction (*II*) are admissible in HK_n^c, $HK4_n^c$, HT_n^c and $HS4_n^c$:*

$$\frac{F \vee G}{G \vee F}\,\mathrm{CD} \qquad \frac{F \supset G}{\neg F \vee G}\,\mathrm{IR} \qquad \frac{F \vee G}{\neg F \supset G}\,\mathrm{II}$$

Proof. The proof is easy and follows immediately from respective derivable formulas of Lemma 2.3.6 and *MP* rule. Only *CD* rule is shown, because the other ones are completely analogous.

1.	$F \vee G$	An assumption of the rule.
2.	$(F \vee G) \supset (G \vee F)$	Lemma 2.3.6, 1a.
3.	$G \vee F$	*MP* rule from 1 and 2.

$\square$

Lemma 2.3.8. *The rules of expansion by conjunction (*E$\wedge_1$ *and* E$\wedge_2$*) and expansion by disjunction (*E$\vee_1$ *and* E$\vee_2$*) are admissible in* HK_n^c, $HK4_n^c$, HT_n^c *and* $HS4_n^c$*:*

$$\frac{F \supset G}{(H \wedge F) \supset (H \wedge G)}\,\mathrm{E}\wedge_1 \qquad \frac{F \supset G}{(F \wedge H) \supset (G \wedge H)}\,\mathrm{E}\wedge_2$$

$$\frac{F \supset G}{(H \vee F) \supset (H \vee G)}\,\mathrm{E}\vee_1 \qquad \frac{F \supset G}{(F \vee H) \supset (G \vee H)}\,\mathrm{E}\vee_2$$

Proof. Once again, only the proof of $E\wedge_1$ is shown. The proofs of admissibility of other rules are completely analogous:

1.	$F \supset G$	An assumption of the rule.
2.	$(H \wedge F) \supset H$	Axiom *2.1*, $\{{}^{H}/_{F}, {}^{F}/_{G}\}$.
3.	$(H \wedge F) \supset F$	Axiom *2.2*, $\{{}^{H}/_{F}, {}^{F}/_{G}\}$.
4.	$(H \wedge F) \supset G$	*Tr* rule from 3 and 1.
5.	$(H \wedge F) \supset (H \wedge G)$	*R*$\wedge$ rule from 2 and 4.

$\square$

Finally, the soundness of the defined sequent calculi with cut can be shown. Once again, in order to prove that, the soundness of respective Hilbert-type calculi is used.

Theorem 2.3.9. *Calculus* GK_n^c*cut is sound. That is, if some sequent* S *is derivable in* GK_n^c*cut, then formula* $\mathrm{Cor}(S)$ *is derivable in* HK_n^c*.*

Proof. Let $\mathcal{D}$ be the derivation tree of some sequent S in GK_n^c*cut*. Now let's show how to construct the derivation of $\mathrm{Cor}(S)$ in HK_n^c. Induction on the height of $\mathcal{D}$ is used.

The induction base. If $\mathrm{h}(\mathcal{D}) = 1$, then S is an axiom $\Gamma, F \to F, \Delta$. If $\Gamma = G_1, \ldots, G_n$, $\Delta = H_1, \ldots, H_m$ and $n, m \geqslant 1$, then $\mathrm{Cor}(S) = (G \wedge F) \supset (F \wedge H)$, where $G = G_1 \wedge \ldots \wedge G_n$ and $H = H_1 \vee \ldots \vee H_m$. The derivation of this formula in HK_n^c is as follows:

1.	$(G \wedge F) \supset F$	Axiom *2.2*, $\{{}^{G}/_{F}, {}^{F}/_{G}\}$.
2.	$F \supset (F \vee H)$	Axiom *3.1*, $\{{}^{F}/_{F}, {}^{H}/_{G}\}$.
3.	$(G{\wedge}F) \supset (F{\vee}H)$	*Tr* rule from 1 and 2.

If $\Gamma = \varnothing$ or $\Delta = \varnothing$, then $\mathrm{Cor}(S)$ is already axiom of HK_n^c (*3.1* and *2.2* respectively). If $\Gamma = \Delta = \varnothing$, then $\mathrm{Cor}(S) = F \supset F$ and this formula is derivable in HK_n^c as shown in Example 1.1.6.

The induction step. Let's assume that it is possible to construct the derivation of $\mathrm{Cor}(S^{'})$ in HK_n^c, if the height of the derivation tree of $S^{'}$ in $GK_n^c cut$ is lower than k. Suppose $\mathrm{h}(\mathcal{D}) = k$. Let's check the bottom-most inference in $\mathcal{D}$.

If rule $(\neg \to)$ is applied, then $\mathcal{D}$ looks like this:

$$\dfrac{\begin{matrix}\mathcal{D}^{'}\\ S_1 = \Gamma \to \Delta, F\end{matrix}}{S = \neg F, \Gamma \to \Delta}\,(\neg \to)$$

According to induction hypothesis, $\vdash_{HK_n^c} \mathrm{Cor}(S_1)$, because $\mathrm{h}(\mathcal{D}^{'}) < k$. If $\Gamma = G_1, \ldots, G_n$, $\Delta = H_1, \ldots, H_m$ and $n, m \geqslant 1$, then $\mathrm{Cor}(S_1) = G \supset (H \vee F)$ and $\mathrm{Cor}(S) = (\neg F \wedge G) \supset H$, where $G = G_1 \wedge \ldots \wedge G_n$ and $H = H_1 \vee \ldots \vee H_m$. The derivation of $\mathrm{Cor}(S)$ in HK_n^c is as follows:

1.	$G \supset (H \vee F)$	Induction hypothesis.
2.	$\neg G \vee H \vee F$	*IR* rule from 1.
3.	$F \vee \neg G \vee H$	*CD* rule from 2.
4.	$\neg(F \vee \neg G) \supset H$	*II* rule from 3.
5.	$(\neg F \wedge \neg\neg G) \supset \neg(F \vee \neg G)$	Lemma 2.3.6, 1d, $\{F/F, \neg G/G\}$.
6.	$(\neg F \wedge \neg\neg G) \supset H$	*Tr* rule from 5 and 4.
7.	$G \supset \neg\neg G$	Axiom *4.2*, $\{G/F\}$.
8.	$(\neg F \wedge G) \supset (\neg F \wedge \neg\neg G)$	$E\wedge_1$ rule from 7.
9.	$(\neg F \wedge G) \supset H$	*Tr* rule from 8 and 6.

If $\Gamma = \varnothing$ and $\Delta \neq \varnothing$, then this derivation can be modified to prove that if $H \vee F$ is derivable, then $\neg F \supset H$ is derivable too.

If $\Delta = \varnothing$ and $\Gamma \neq \varnothing$, then $\mathrm{Cor}(S_1) = G \supset F$, $\mathrm{Cor}(S) = \neg(\neg F \wedge G)$ and the derivation of $\mathrm{Cor}(S)$ is as follows:

1.	$G \supset F$	Induction hypothesis.
2.	$\neg G \vee F$	*IR* rule from 1.
3.	$F \vee \neg G$	*CD* rule from 2.
4.	$(\neg F \wedge \neg\neg G) \supset \neg(F \vee \neg G)$	Lemma 2.3.6, 1d, $\{F/F, \neg G/G\}$.
5.	$G \supset \neg\neg G$	Axiom *4.2*, $\{G/F\}$.
6.	$(\neg F \wedge G) \supset (\neg F \wedge \neg\neg G)$	$E\wedge_1$ rule from 5.
7.	$(\neg F \wedge G) \supset \neg(F \vee \neg G)$	*Tr* rule from 6 and 4.
8.	$\neg\neg(F \vee \neg G) \supset \neg(\neg F \wedge G)$	$R\neg$ rule from 7.
9.	$(F \vee \neg G) \supset \neg\neg(F \vee \neg G)$	Axiom *4.2*, $\{F \vee \neg G/F\}$.
10.	$(F \vee \neg G) \supset \neg(\neg F \wedge G)$	*Tr* rule from 9 and 8.
11.	$\neg(\neg F \wedge G)$	*MP* rule from 3 and 10.

If $\Gamma = \Delta = \varnothing$, then $\mathrm{Cor}(S_1) = F$, $\mathrm{Cor}(S) = \neg\neg F$ and the derivation of $\mathrm{Cor}(S)$ is obvious:

1.	F	Induction hypothesis.
2.	$F \supset \neg\neg F$	Axiom *4.2*.
3.	$\neg\neg F$	*MP* rule from 1 and 2.

If rule $(\wedge \to)$ is applied, then $\mathcal{D}$ looks like this:

$$\dfrac{\begin{array}{c}\mathcal{D}^{'}\\ S_1 = F_1, F_2, \Gamma \to \Delta\end{array}}{S = F_1 \wedge F_2, \Gamma \to \Delta}\ (\wedge \to)$$

In this case there is nothing to prove, because in all the cases $\mathrm{Cor}(S_1) = \mathrm{Cor}(S)$. Indeed, if $\Gamma \neq \varnothing$ and $\Delta \neq \varnothing$, then $\mathrm{Cor}(S_1) = \mathrm{Cor}(S) = (F_1 \wedge F_2 \wedge G) \supset H$, if $\Gamma = \varnothing$ and $\Delta \neq \varnothing$, then $\mathrm{Cor}(S_1) = \mathrm{Cor}(S) = (F_1 \wedge F_2) \supset H$, if $\Delta = \varnothing$ and $\Gamma \neq \varnothing$, then $\mathrm{Cor}(S_1) = \mathrm{Cor}(S) = \neg(F_1 \wedge F_2 \wedge G)$ and if $\Gamma = \Delta = \varnothing$, then $\mathrm{Cor}(S_1) = \mathrm{Cor}(S) = \neg(F_1 \wedge F_2)$. Once again, here $\Gamma = G_1, \ldots, G_n$, $\Delta = H_1, \ldots, H_m$, if $n \geqslant 1$, then $G = G_1 \wedge \ldots \wedge G_n$ and if $m \geqslant 1$, then $H = H_1 \vee \ldots \vee H_m$.

If rule $(\to \wedge)$ is applied, then $\mathcal{D}$ looks like this:

$$\dfrac{\begin{array}{c}\mathcal{D}^{'}\\ S_1 = \Gamma \to \Delta, F_1\end{array} \qquad \begin{array}{c}\mathcal{D}^{''}\\ S_2 = \Gamma \to \Delta, F_2\end{array}}{S = \Gamma \to \Delta, F_1 \wedge F_2}\ (\neg \to)$$

According to induction hypothesis, $\vdash_{HK_n^c} \mathrm{Cor}(S_1)$, because $\mathrm{h}(\mathcal{D}^{'}) < k$ and $\vdash_{HK_n^c} \mathrm{Cor}(S_2)$, because $\mathrm{h}(\mathcal{D}^{''}) < k$. If $\Gamma = G_1, \ldots, G_n$, $\Delta = H_1, \ldots, H_m$ and $n, m \geqslant 1$, then $\mathrm{Cor}(S_1) = G \supset (H \vee F_1)$, $\mathrm{Cor}(S_2) = G \supset (H \vee F_2)$ and $\mathrm{Cor}(S) = G \supset \big(H \vee (F_1 \wedge F_2)\big)$, where $G = G_1 \wedge \ldots \wedge G_n$ and $H = H_1 \vee \ldots \vee H_m$. The derivation of $\mathrm{Cor}(S)$ in HK_n^c is as follows:

1.	$G \supset (H \vee F_1)$	Induction hypothesis.
2.	$(H \vee F_1) \supset (\neg H \supset F_1)$	Lemma 2.3.6, 1c, $\{{}^{H}/_{F}, {}^{F_1}/_{G}\}$.
3.	$G \supset (\neg H \supset F_1)$	*Tr* rule from 1 and 2.
4.	$G \supset (H \vee F_2)$	Induction hypothesis.
5.	$(H \vee F_2) \supset (\neg H \supset F_2)$	Lemma 2.3.6, 1c, $\{{}^{H}/_{F}, {}^{F_2}/_{G}\}$.
6.	$G \supset (\neg H \supset F_2)$	*Tr* rule from 4 and 5.
7.	$(\neg H \supset F_1) \supset \Big((\neg H \supset F_2) \supset \big(\neg H \supset (F_1 \wedge F_2)\big)\Big)$	Axiom *2.3*, $\{{}^{\neg H}/_{F}, {}^{F_1}/_{G}, {}^{F_2}/_{H}\}$.
8.	$G \supset \Big((\neg H \supset F_2) \supset \big(\neg H \supset (F_1 \wedge F_2)\big)\Big)$	*Tr* rule from 3 and 7.
9.	$G \supset \big(\neg H \supset (F_1 \wedge F_2)\big)$	$R{\supset}$ rule from 8 and 6.
10.	$\big(\neg H \supset (F_1 \wedge F_2)\big) \supset \big(\neg\neg H \vee (F_1 \wedge F_2)\big)$	Lemma 2.3.6, 1b, $\{{}^{\neg H}/_{F}, {}^{F_1 \wedge F_2}/_{G}\}$.
11.	$G \supset \big(\neg\neg H \vee (F_1 \wedge F_2)\big)$	*Tr* rule from 9 and 10.
12.	$\neg\neg H \supset H$	Axiom *4.3*, $\{{}^{H}/_{F}\}$.
13.	$\big(\neg\neg H \vee (F_1 \wedge F_2)\big) \supset \big(H \vee (F_1 \wedge F_2)\big)$	$E{\vee}_2$ rule from 12.
14.	$G \supset \big(H \vee (F_1 \wedge F_2)\big)$	*Tr* rule from 11 and 13.

If $\Gamma = \varnothing$ and $\Delta \neq \varnothing$, then this derivation can be modified to prove that if $H \vee F_1$ and $H \vee F_2$ are derivable, then $H \vee (F_1 \wedge F_2)$ is derivable too.

If $\Delta = \varnothing$ and $\Gamma \neq \varnothing$, then $\mathrm{Cor}(S_1) = G \supset F_1$, $\mathrm{Cor}(S_2) = G \supset F_2$, $\mathrm{Cor}(S) = G \supset (F_1 \wedge F_2)$ and the derivation of $\mathrm{Cor}(S)$ is as follows:

1. $G \supset F_1$ Induction hypothesis.
2. $G \supset F_2$ Induction hypothesis.
3. $G \supset (F_1 \wedge F_2)$ *R*$\wedge$ rule from 1 and 2.

If $\Gamma = \Delta = \varnothing$, then $\mathrm{Cor}(S_1) = F_1$, $\mathrm{Cor}(S_2) = F_2$, $\mathrm{Cor}(S) = F_1 \wedge F_2$ and the derivation of $\mathrm{Cor}(S)$ is as follows:

1. F_1 Induction hypothesis.
2. F_2 Induction hypothesis.
3. $F_1 \supset F_1$ As in Example 1.1.6.
4. $F_2 \supset (F_1 \supset F_2)$ Axiom *1.1*, $\{{}^{F_2}/_{F}, {}^{F_1}/_{G}\}$.
5. $F_1 \supset F_2$ *MP* rule from 2 and 4.
6. $F_1 \supset (F_1 \wedge F_2)$ *R*$\wedge$ rule from 3 and 5.
7. $F_1 \wedge F_2$ *MP* rule from 1 and 6.

The cases of other logical rules — $(\rightarrow \neg)$, $(\vee \rightarrow)$, $(\rightarrow \vee)$, $(\supset\rightarrow)$, $(\rightarrow\supset)$ — are completely analogous to these three cases analysed here.

If rule (cut F) is applied, then $\mathcal{D}$ looks like this:

$$\frac{\begin{array}{cc} \mathcal{D}' & \mathcal{D}'' \\ S_1 = \Gamma \rightarrow \Delta, F & S_2 = F, \Pi \rightarrow \Lambda \end{array}}{S = \Gamma, \Pi \rightarrow \Delta, \Lambda}\ (\text{cut } F)$$

According to induction hypothesis, $\vdash_{HK_n^c} \mathrm{Cor}(S_1)$, because $\mathrm{h}(\mathcal{D}') < k$ and $\vdash_{HK_n^c} \mathrm{Cor}(S_2)$, because $\mathrm{h}(\mathcal{D}'') < k$. Now if $\Gamma = G_{1,1}, \ldots, G_{1,n_1}$, where $n_1 \geqslant 1$, $\Pi = G_{2,1}, \ldots, G_{2,n_2}$, where $n_2 \geqslant 1$, $\Delta = H_{1,1}, \ldots, H_{1,m_1}$, where $m_1 \geqslant 1$, and $\Lambda = H_{2,1}, \ldots, H_{2,m_2}$, where $m_2 \geqslant 1$, then $\mathrm{Cor}(S_1) = G_1 \supset (H_1 \vee F)$, $\mathrm{Cor}(S_2) = (F \wedge G_2) \supset H_2$ and $\mathrm{Cor}(S) = (G_1 \wedge G_2) \supset (H_1 \vee H_2)$, where $G_1 = G_{1,1} \wedge \ldots \wedge G_{1,n_1}$, $G_2 = G_{2,1} \wedge \ldots \wedge G_{2,n_2}$, $H_1 = H_{1,1} \vee \ldots \vee H_{1,m_1}$ and $H_2 = H_{2,1} \vee \ldots \vee H_{2,m_2}$. The derivation of $\mathrm{Cor}(S)$ in HK_n^c is as follows:

1. $G_1 \supset (H_1 \vee F)$ Induction hypothesis.
2. $\neg G_1 \vee H_1 \vee F$ *IR* rule from 1.
3. $\neg(\neg G_1 \vee H_1) \supset F$ *II* rule from 2.
4. $(F \wedge G_2) \supset H_2$ Induction hypothesis.
5. $\neg(F \wedge G_2) \vee H_2$ *IR* rule from 4.
6. $\neg(F \wedge G_2) \supset (\neg F \vee \neg G_2)$ Lemma 2.3.6, 1e, $\{{}^{F}/_{F}, {}^{G_2}/_{G}\}$.
7. $\neg F \vee \neg G_2 \vee H_2$ *SD* rule from 5 and 6.
8. $\neg\neg F \supset (\neg G_2 \vee H_2)$ *II* rule from 7.
9. $F \supset \neg\neg F$ Axiom *4.2*.
10. $F \supset (\neg G_2 \vee H_2)$ *Tr* rule from 9 and 8.
11. $\neg(\neg G_1 \vee H_1) \supset (\neg G_2 \vee H_2)$ *Tr* rule from 3 and 10.
12. $\neg\neg(\neg G_1 \vee H_1) \vee \neg G_2 \vee H_2$ *IR* rule from 11.
13. $\neg\neg(\neg G_1 \vee H_1) \supset (\neg G_1 \vee H_1)$ Axiom *4.3*, $\{{}^{\neg G_1 \vee H_1}/_{F}\}$.
14. $\neg G_1 \vee H_1 \vee \neg G_2 \vee H_2$ *SD* rule from 12 and 13.

15. $(H_1 \vee \neg G_2) \supset (\neg G_2 \vee H_1)$ — Lemma 2.3.6, 1a, $\{{}^{H_1}/_F, {}^{\neg G_2}/_G\}$.
16. $(\neg G_1 \vee H_1 \vee \neg G_2) \supset (\neg G_1 \vee \neg G_2 \vee H_1)$ — $E\vee_1$ rule from 15.
17. $\neg G_1 \vee \neg G_2 \vee H_1 \vee H_2$ — *SD* rule from 14 and 16.
18. $\neg(\neg G_1 \vee \neg G_2) \supset (H_1 \vee H_2)$ — *II* rule from 17.
19. $(\neg\neg G_1 \wedge \neg\neg G_2) \supset \neg(\neg G_1 \vee \neg G_2)$ — Lemma 2.3.6, 1d, $\{{}^{\neg G_1}/_F, {}^{\neg G_2}/_G\}$.
20. $(\neg\neg G_1 \wedge \neg\neg G_2) \supset (H_1 \vee H_2)$ — Transitivity rule from 19 and 18.
21. $G_1 \supset \neg\neg G_1$ — Axiom *4.2*, $\{{}^{G_1}/_F\}$.
22. $(G_1 \wedge \neg\neg G_2) \supset (\neg\neg G_1 \wedge \neg\neg G_2)$ — $E\wedge_2$ rule from 21.
23. $(G_1 \wedge \neg\neg G_2) \supset (H_1 \vee H_2)$ — *Tr* rule from 22 and 20.
24. $G_2 \supset \neg\neg G_2$ — Axiom *4.2*, $\{{}^{G_2}/_F\}$.
25. $(G_1 \wedge G_2) \supset (G_1 \wedge \neg\neg G_2)$ — $E\wedge_1$ rule from 24.
26. $(G_1 \wedge G_2) \supset (H_1 \vee H_2)$ — *Tr* rule from 25 and 23.

To completely show that an application of (cut F) rule can be replaced by derivation in HK_n^c, all the possible combinations of Γ, Δ, Π and Λ being empty or not must be analysed. There are 16 cases in total. One of them, when all the sets are not empty, is presented above. Let's analyse two more cases

When $\Gamma \neq \varnothing$, $\Delta \neq \varnothing$, but $\Pi = \Lambda = \varnothing$, then $\mathrm{Cor}(S_1) = G_1 \supset (H_1 \vee F)$, $\mathrm{Cor}(S_2) = \neg F$ and $\mathrm{Cor}(S) = G_1 \supset H_1$. The derivation of $\mathrm{Cor}(S)$ in HK_n^c is as follows:

1. $G_1 \supset (H_1 \vee F)$ — Induction hypothesis.
2. $\neg G_1 \vee H_1 \vee F$ — *IR* rule from 1.
3. $\neg(\neg G_1 \vee H_1) \supset F$ — *II* rule from 2.
4. $\neg F \supset \neg\neg(\neg G_1 \vee H_1)$ — $R\neg$ rule from 3.
5. $\neg\neg(\neg G_1 \vee H_1) \supset (\neg G_1 \vee H_1)$ — Axiom *4.3*, $\{{}^{\neg G_1 \vee H_1}/_F\}$.
6. $\neg F \supset (\neg G_1 \vee H_1)$ — *Tr* rule from 4 and 5.
7. $\neg F$ — Induction hypothesis.
8. $\neg G_1 \vee H_1$ — *MP* rule from 7 and 6.
9. $\neg\neg G_1 \supset H_1$ — *II* rule from 8.
10. $G_1 \supset \neg\neg G_1$ — Axiom *4.2*, $\{{}^{G_1}/_F\}$.
11. $G_1 \supset H_1$ — *Tr* rule from 10 and 9.

The next case is when $\Gamma = \Delta = \Pi = \Lambda = \varnothing$. In this case, $\mathrm{Cor}(S_1) = F$, $\mathrm{Cor}(S_2) = \neg F$ and $\mathrm{Cor}(S) = p \wedge \neg p$ for some propositional variable p. The derivation of $\mathrm{Cor}(S)$ in HK_n^c is as follows:

1. F — Induction hypothesis.
2. $\neg F$ — Induction hypothesis.
3. $\big(\neg(p \wedge \neg p) \supset F\big) \supset \big(\neg F \supset \neg\neg(p \wedge \neg p)\big)$ — Axiom *4.1*, $\{{}^{\neg(p\wedge\neg p)}/_F, {}^{F}/_G\}$.
4. $F \supset \big(\neg(p \wedge \neg p) \supset F\big)$ — Axiom *1.1*, $\{{}^{F}/_F, {}^{\neg(p\wedge\neg p)}/_G\}$.
5. $F \supset \big(\neg F \supset \neg\neg(p \wedge \neg p)\big)$ — *Tr* rule from 4 and 3.
6. $\neg F \supset \neg\neg(p \wedge \neg p)$ — *MP* rule from 1 and 5.
7. $\neg\neg(p \wedge \neg p) \supset (p \wedge \neg p)$ — Axiom *4.3*, $\{{}^{p\wedge\neg p}/_F\}$.
8. $\neg F \supset (p \wedge \neg p)$ — *Tr* rule from 6 and 7.
9. $p \wedge \neg p$ — *MP* rule from 2 and 8.

This case actually proves one well known rule of logic: if for some formula F both F and $\neg F$ are derivable in some system, then any formula is derivable in that system. In fact $p \wedge \neg p$ is not valid in any considered logic and therefore, not derivable in any considered calculus. According to this reasoning, empty sequent $\rightarrow$ is not derivable in $GK_n^c cut$.

All the other cases are similar to the ones, that are already presented.

If rule $(\rightarrow \Box^c)$ is applied, then $\mathcal{D}$ looks like this:

$$\dfrac{\begin{matrix}\mathcal{D}' \\ S_1 = \Gamma_2 \rightarrow F\end{matrix}}{S = \Gamma_1, \Box_* \Gamma_2 \rightarrow \Delta, \Box_c F}\ (\rightarrow \Box^c)$$

According to induction hypothesis, $\vdash_{HK_n^c} \mathrm{Cor}(S_1)$, because $\mathrm{h}(\mathcal{D}') < k$. Now if $\Gamma_1 = H_{1,1}, \ldots, H_{1,m_1}$, where $m_1 \geqslant 1$, $\Gamma_2 = G_1, \ldots, G_n$, where $n \geqslant 1$ and $\Delta = H_{2,1}, \ldots, H_{2,m_2}$, where $m_2 \geqslant 1$, then $\mathrm{Cor}(S_1) = (G_1 \wedge \ldots \wedge G_n) \supset F$ and $\mathrm{Cor}(S) = (H_1 \wedge \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset (H_2 \vee \Box_c F)$, where $H_1 = H_{1,1} \wedge \ldots \wedge H_{1,m_1}$ and $H_2 = H_{2,1} \vee \ldots \vee H_{2,m_2}$. The derivation of $\mathrm{Cor}(S)$ in HK_n^c is as follows:

1.	$(G_1 \wedge \ldots \wedge G_n) \supset F$	Induction hypothesis.
2.	$\Box_c\big((G_1 \wedge \ldots \wedge G_n) \supset F\big)$	NG_c rule from 1.
3.	$\Box_c(G_1 \wedge \ldots \wedge G_n) \supset \Box_c F$	K_c rule from 2.
4.	$(\Box_c G_1 \wedge \ldots \wedge \Box_c G_n) \supset \Box_c(G_1 \wedge \ldots \wedge G_n)$	Lemma 2.3.6, 1f.
5.	$(\Box_c G_1 \wedge \ldots \wedge \Box_c G_n) \supset \Box_c F$	*Tr* rule from 4 and 3.
6.	$(\Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset (\Box_c G_1 \wedge \ldots \wedge \Box_c G_n)$	Lemma 2.3.6, 1g.
7.	$(\Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset \Box_c F$	*Tr* rule from 6 and 5.
8.	$(H_1 \wedge \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset (\Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n)$	Axiom *2.2*, $\{H_1/F, \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n/G\}$.
9.	$(H_1 \wedge \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset \Box_c F$	*Tr* rule from 8 and 7.
10.	$\Box_c F \supset (H_2 \vee \Box_c F)$	Axiom *3.2*, $\{H_2/F, \Box_c F/G\}$.
11.	$(H_1 \wedge \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset (H_2 \vee \Box_c F)$	*Tr* rule from 9 and 10.

This derivation can be easily altered to deal with cases, where Γ_1, Δ or both of them are empty. For example, if Γ_1 and Δ are empty, then the derivation is the same, but terminates in step 7.

If Γ_2 is empty (but neither Γ_1 nor Δ is), then $\mathrm{Cor}(S_1) = F$ and $\mathrm{Cor}(S) = H_1 \supset (H_2 \vee \Box_c F)$. The derivation of $\mathrm{Cor}(S)$ in HK_n^c is as follows:

1.	F	Induction hypothesis.
2.	$\Box_c F$	NG_c rule from 1.
3.	$\Box_c F \supset (H_1 \supset \Box_c F)$	Axiom *1.1*, $\{\Box_c F/F, H_1/G\}$.
4.	$H_1 \supset \Box_c F$	*MP* rule from 2 and 3.
5.	$\Box_c F \supset (H_2 \vee \Box_c F)$	Axiom *3.2*, $\{H_2/F, \Box_c F/G\}$.
6.	$H_1 \supset (H_2 \vee \Box_c F)$	*Tr* rule from 4 and 5.

This derivation can also be easily modified to deal with cases, where Γ_1 or Δ is empty.

The application of $(\rightarrow \Box_l)$ is changed to Hilbert-type derivation in a similar way as the application of rule $(\rightarrow \Box^c)$. This completes the proof that if some sequent S is derivable in $GK_n^c cut$, then formula $\mathrm{Cor}(S)$ is derivable in HK_n^c. $\Box$

Theorem 2.3.10. *Calculus $GK4_n^c cut$ is sound. That is, if some sequent S is derivable in $GK4_n^c cut$, then formula $\mathrm{Cor}(S)$ is derivable in $HK4_n^c$.*

Proof. Let $\mathcal{D}$ be the derivation tree of some sequent S in $GK4_n^c cut$. Now let's show how to construct the derivation of $\mathrm{Cor}(S)$ in $HK4_n^c$. Once again induction on the height of $\mathcal{D}$ is used.

In fact the proof is very similar to the proof of Lemma 2.3.9. The only difference is how the cases of application of rules $(\rightarrow \Box_l)$ and $(\rightarrow \Box^c)$ are dealt with.

If rule $(\rightarrow \Box^c)$ is applied, then $\mathcal{D}$ looks like this:

$$\dfrac{\begin{array}{c}\mathcal{D}' \\ S_1 = \Gamma_2, \Box_* \Gamma_2 \rightarrow F\end{array}}{S = \Gamma_1, \Box_* \Gamma_2 \rightarrow \Delta, \Box_c F}\,(\rightarrow \Box^c)$$

According to induction hypothesis, $\vdash_{HK_n^c} \mathrm{Cor}(S_1)$, because $\mathrm{h}(\mathcal{D}') < k$. Now if $\Gamma_1 = H_{1,1}, \ldots, H_{1,m_1}$, where $m_1 \geqslant 1$, $\Gamma_2 = G_1, \ldots, G_n$, where $n \geqslant 1$ and $\Delta = H_{2,1}, \ldots, H_{2,m_2}$, where $m_2 \geqslant 1$, then $\mathrm{Cor}(S_1) = (G_1 \wedge \ldots \wedge G_n \wedge \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset F$ and $\mathrm{Cor}(S) = (H_1 \wedge \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset (H_2 \vee \Box_c F)$, where $H_1 = H_{1,1} \wedge \ldots \wedge H_{1,m_1}$ and $H_2 = H_{2,1} \vee \ldots \vee H_{2,m_2}$. The derivation of $\mathrm{Cor}(S)$ in HK_n^c is as follows:

1.	$(G_1 \wedge \ldots \wedge G_n \wedge \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset F$	Induction hypothesis.
2.	$\Box_c\big((G_1 \wedge \ldots \wedge G_n \wedge \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset F\big)$	NG_c rule from 1.
3.	$\Box_c(G_1 \wedge \ldots \wedge G_n \wedge \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset \Box_c F$	K_c rule from 2.
4.	$(\Box_c G_1 \wedge \ldots \wedge \Box_c G_n \wedge \Box_c \Box_{l_1} G_1 \wedge \ldots \wedge \Box_c \Box_{l_n} G_n) \supset \Box_c(G_1 \wedge \ldots \wedge G_n \wedge \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n)$	Lemma 2.3.6, 1f.
5.	$(\Box_c G_1 \wedge \ldots \wedge \Box_c G_n \wedge \Box_c \Box_{l_1} G_1 \wedge \ldots \wedge \Box_c \Box_{l_n} G_n) \supset \Box_c F$	*Tr* rule from 4 and 3.
6.	$(\Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset (\Box_c G_1 \wedge \ldots \wedge \Box_c G_n)$	Lemma 2.3.6, 1g.
7.	$(\Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset (\Box_{l_1} \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} \Box_{l_n} G_n)$	Lemma 2.3.6, 2a.
8.	$(\Box_{l_1} \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} \Box_{l_n} G_n) \supset (\Box_c \Box_{l_1} G_1 \wedge \ldots \wedge \Box_c \Box_{l_n} G_n)$	Lemma 2.3.6, 1g.
9.	$(\Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset (\Box_c \Box_{l_1} G_1 \wedge \ldots \wedge \Box_c \Box_{l_n} G_n)$	*Tr* rule from 7 and 8.
10.	$(\Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset (\Box_c G_1 \wedge \ldots \wedge \Box_c G_n \wedge \Box_c \Box_{l_1} G_1 \wedge \ldots \wedge \Box_c \Box_{l_n} G_n)$	$R\wedge$ rule from 6 and 9.
11.	$(\Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset \Box_c F$	*Tr* rule from 10 and 5.
12.	$(H_1 \wedge \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset (\Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n)$	Axiom *2.2*, $\{{}^{H_1}/_{F}, {}^{\Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n}/_{G}\}$.
13.	$(H_1 \wedge \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset \Box_c F$	*Tr* rule from 12 and 11.
14.	$\Box_c F \supset (H_2 \vee \Box_c F)$	Axiom *3.2*, $\{{}^{H_2}/_{F}, {}^{\Box_c F}/_{G}\}$.
15.	$(H_1 \wedge \Box_{l_1} G_1 \wedge \ldots \wedge \Box_{l_n} G_n) \supset (H_2 \vee \Box_c F)$	*Tr* rule from 13 and 14.

This derivation can be easily altered to deal with cases, where Γ_1, Δ or both of them are empty.

If Γ_2 is empty, then $\mathrm{Cor}(S_1) = F$ and this case is proved in the same way as analogous case of the proof of Lemma 2.3.9.

The application of $(\rightarrow \Box_l)$ is changed to Hilbert-type derivation in a similar way as the application of rule $(\rightarrow \Box^c)$. $\Box$

Theorem 2.3.11. *Calculus $GT_n^c cut$ is sound. That is, if some sequent S is derivable in $GT_n^c cut$, then formula* $\mathrm{Cor}(S)$ *is derivable in HT_n^c.*

Proof. All the rules and axioms of calculus $GT_n^c cut$ are the same as the ones of calculus $GK_n^c cut$, except that calculus $GT_n^c cut$ has rule $(\Box_l \rightarrow)$, which is not part of calculus $GK_n^c cut$. Therefore, the proof of this lemma is the same as the proof of Lemma 2.3.9, except that the case of the rule $(\Box_l \rightarrow)$ must be analysed additionally.

If rule $(\Box_l \rightarrow)$ is applied, then $\mathcal{D}$ looks like this:

$$\dfrac{\begin{array}{c}\mathcal{D}' \\ S_1 = F, \Box_l F, \Gamma \rightarrow \Delta\end{array}}{S = \Box_l F, \Gamma \rightarrow \Delta}\ (\Box_l \rightarrow)$$

According to induction hypothesis, $\vdash_{HK_n^c} \mathrm{Cor}(S_1)$, because $\mathrm{h}(\mathcal{D}') < k$. If $\Gamma = G_1, \ldots, G_n$, $\Delta = H_1, \ldots, H_m$ and $n, m \geqslant 1$, then $\mathrm{Cor}(S_1) = (F \wedge \Box_l F \wedge G) \supset H$ and $\mathrm{Cor}(S) = (\Box_l F \wedge G) \supset H$, where $G = G_1 \wedge \ldots \wedge G_n$ and $H = H_1 \vee \ldots \vee H_m$. The derivation of $\mathrm{Cor}(S)$ in HK_n^c is as follows:

1.	$(F \wedge \Box_l F \wedge G) \supset H$	Induction hypothesis.
2.	$\Box_l F \supset F$	Axiom *(T_l)*.
3.	$\Box_l F \supset \Box_l F$	As in Example 1.1.6.
4.	$\Box_l F \supset (F \wedge \Box_l F)$	$R\wedge$ rule from 2 and 3.
5.	$(\Box_l F \wedge G) \supset (F \wedge \Box_l F \wedge G)$	$E\wedge_2$ rule from 4.
6.	$(\Box_l F \wedge G) \supset H$	Tr rule from 5 and 1.

This derivation can be easily altered to deal with case, where $\Gamma = \varnothing$ and $\Delta \neq \varnothing$.

If $\Gamma \neq \varnothing$, but $\Delta = \varnothing$, then $\mathrm{Cor}(S_1) = \neg(F \wedge \Box_l F \wedge G)$, $\mathrm{Cor}(S) = \neg(\Box_l F \wedge G)$ and the derivation of $\mathrm{Cor}(S)$ is as follows:

1.	$\neg(F \wedge \Box_l F \wedge G)$	Induction hypothesis.
2.	$(\Box_l F \wedge G) \supset (F \wedge \Box_l F \wedge G)$	As in the previous derivation from 2 to 5.
3.	$\neg(F \wedge \Box_l F \wedge G) \supset \neg(\Box_l F \wedge G)$	$R\neg$ rule from 3.
4.	$\neg(\Box_l F \wedge G)$	MP rule from 1 and 3.

This derivation can also be modified to deal with case, where $\Gamma = \Delta = \varnothing$. $\Box$

Theorem 2.3.12. *Calculus $GS4_n^c cut$ is sound. That is, if some sequent S is derivable in $GS4_n^c cut$, then formula* $\mathrm{Cor}(S)$ *is derivable in $HS4_n^c$.*

Proof. Calculus $GS4_n^c cut$ is the same as calculus $GT_n^c cut$, except that rules $(\to \Box_l)$ and $(\to \Box^c)$ are different. Therefore, the proof of this lemma is the same as the proof of Lemma 2.3.11, except the cases of the mentioned rules, that are provided further.

If rule $(\to \Box^c)$ is applied, then $\mathcal{D}$ looks like this:

$$\cfrac{\begin{matrix}\mathcal{D}'\\ S_1 = \Box_* \Gamma_2 \to F\end{matrix}}{S = \Gamma_1, \Box_* \Gamma_2 \to \Delta, \Box_c F}\ (\to \Box^c)$$

According to induction hypothesis, $\vdash_{HK_n^c} \mathrm{Cor}(S_1)$, because $\mathrm{h}(\mathcal{D}') < k$. Now if $\Gamma_1 = H_{1,1}, \ldots, H_{1,m_1}$, where $m_1 \geqslant 1$, $\Gamma_2 = G_1, \ldots, G_n$, where $n \geqslant 1$ and $\Delta = H_{2,1}, \ldots, H_{2,m_2}$, where $m_2 \geqslant 1$, then $\mathrm{Cor}(S_1) = (\Box_{l_1} G_1 \land \ldots \land \Box_{l_n} G_n) \supset F$ and $\mathrm{Cor}(S) = (H_1 \land \Box_{l_1} G_1 \land \ldots \land \Box_{l_n} G_n) \supset (H_2 \lor \Box_c F)$, where $H_1 = H_{1,1} \land \ldots \land H_{1,m_1}$ and $H_2 = H_{2,1} \lor \ldots \lor H_{2,m_2}$. The derivation of $\mathrm{Cor}(S)$ in HK_n^c is as follows:

1.	$(\Box_{l_1} G_1 \land \ldots \land \Box_{l_n} G_n) \supset F$	Induction hypothesis.
2.	$\Box_c\big((\Box_{l_1} G_1 \land \ldots \land \Box_{l_n} G_n) \supset F\big)$	NG_c rule from 1.
3.	$\Box_c(\Box_{l_1} G_1 \land \ldots \land \Box_{l_n} G_n) \supset \Box_c F$	K_c rule from 2.
4.	$(\Box_c \Box_{l_1} G_1 \land \ldots \land \Box_c \Box_{l_n} G_n) \supset \Box_c(\Box_{l_1} G_1 \land \ldots \land \Box_{l_n} G_n)$	Lemma 2.3.6, 1f.
5.	$(\Box_c \Box_{l_1} G_1 \land \ldots \land \Box_c \Box_{l_n} G_n) \supset \Box_c F$	*Tr* rule from 4 and 3.
6.	$(\Box_{l_1} G_1 \land \ldots \land \Box_{l_n} G_n) \supset (\Box_{l_1} \Box_{l_1} G_1 \land \ldots \land \Box_{l_n} \Box_{l_n} G_n)$	Lemma 2.3.6, 2a.
7.	$(\Box_{l_1} \Box_{l_1} G_1 \land \ldots \land \Box_{l_n} \Box_{l_n} G_n) \supset$ $(\Box_c \Box_{l_1} G_1 \land \ldots \land \Box_c \Box_{l_n} G_n)$	Lemma 2.3.6, 1g.
8.	$(\Box_{l_1} G_1 \land \ldots \land \Box_{l_n} G_n) \supset (\Box_c \Box_{l_1} G_1 \land \ldots \land \Box_c \Box_{l_n} G_n)$	*Tr* rule from 6 and 7.
9.	$(\Box_{l_1} G_1 \land \ldots \land \Box_{l_n} G_n) \supset \Box_c F$	*Tr* rule from 8 and 5.
10.	$(H_1 \land \Box_{l_1} G_1 \land \ldots \land \Box_{l_n} G_n) \supset (\Box_{l_1} G_1 \land \ldots \land \Box_{l_n} G_n)$	Axiom *2.2*, $\{{}^{H_1}/_F, {}^{\Box_{l_1} G_1 \land \ldots \land \Box_{l_n} G_n}/_G\}$.
11.	$(H_1 \land \Box_{l_1} G_1 \land \ldots \land \Box_{l_n} G_n) \supset \Box_c F$	*Tr* rule from 10 and 9.
12.	$\Box_c F \supset (H_2 \lor \Box_c F)$	Axiom *3.2*, $\{{}^{H_2}/_F, {}^{\Box_c F}/_G\}$.
13.	$(H_1 \land \Box_{l_1} G_1 \land \ldots \land \Box_{l_n} G_n) \supset (H_2 \lor \Box_c F)$	*Tr* rule from 11 and 12.

This derivation can be easily altered to deal with cases, where Γ_1, Δ or both of them are empty.

If Γ_2 is empty, then $\mathrm{Cor}(S_1) = F$ and this case is proved in the same way as analogous case of the proof of Lemma 2.3.9.

The application of $(\to \Box_l)$ is changed to Hilbert-type derivation in a similar way as the application of rule $(\to \Box^c)$. $\Box$

2.4 Gentzen-type Calculi without Cut

The previous section shows, that calculi $GK_n^c cut$, $GK4_n^c cut$, $GT_n^c cut$ and $GS4_n^c cut$ are sound and complete, however they are not terminating. One of the reasons is that it is not possible to go through all the possible rule applications of any sequent. The main problem lies in the rule (cut F). It is not possible to examine all the possible cut formulas. In some cases this problem is solved by restricting the set of possible cut formulas. However, for the discussed calculi, the cut rule can be eliminated completely. This makes sequent calculus for K_n^c terminating and takes other sequent calculi one step closer to the finite derivation search.

Definition 2.4.1. *Gentzen-type calculi without cut for multimodal logics with central agent axiom K_n^c, $K4_n^c$, T_n^c and $S4_n^c$ (respectively, GK_n^c, $GK4_n^c$, GT_n^c and $GS4_n^c$) are obtained from respective Gentzen-type calculi with cut $GK_n^c cut$, $GK4_n^c cut$, $GT_n^c cut$ and $GS4_n^c cut$ by removing the cut rule.*

In this dissertation cut-elimination is proved using invertibility of logical and reflexivity rules and admissibility of weakening and contraction. Alternative method can be found in [2, 51].

Let's start with the admissibility of weakening.

Lemma 2.4.2. *The structural rules of weakening are admissible in sequent calculi GK_n^c, $GK4_n^c$, GT_n^c and $GS4_n^c$.*

Proof. Analogously to the proof of Lemma 1.4.8. $\square$

Next, the invertibility of logical rules and rule $(\Box_l \to)$.

Lemma 2.4.3. *Logical rules are invertible in calculi GK_n^c, $GK4_n^c$, GT_n^c and $GS4_n^c$.*

Proof. Analogously to the proof of Lemma 1.4.11 or 2.3.3 using Lemma 2.4.2. $\square$

Lemma 2.4.4. *Rule $(\Box_l \to)$ is invertible in reflexive multimodal calculi $GK4_n^c$ and $GS4_n^c$.*

Proof. Direct corollary of Lemma 2.4.2, analogously to the proof of Lemma 1.4.12. $\square$

Finally the admissibility of contraction structural rules.

Lemma 2.4.5. *The structural rules of contraction are admissible in calculi GK_n^c, $GK4_n^c$, GT_n^c and $GS4_n^c$.*

Proof. Analogously to the proof of Lemma 1.4.13 using Lemmas 2.4.2, 2.4.3 and 2.4.4. □

In the derivation search trees one or more applications of weakening or contraction structural rules are denoted by double line.

Now, cut elimination can be proved.

Theorem 2.4.6 (Cut elimination for K_n^c). *Sequent is derivable in $GK_n^c cut$ iff it is derivable in GK_n^c.*

Proof. If sequent is derivable in GK_n^c, then obviously it is derivable in $GK_n^c cut$.

To prove the other side let's analyse only those derivation trees in $GK_n^c cut$, that have only one application of (cut F) and it is the bottom-most inference. If the application of cut rule can be eliminated from such derivation trees, then inductively it can be eliminated from any derivation tree in $GK_n^c cut$ to obtain the derivation tree in GK_n^c.

Let's say $\mathcal{D}$ is a derivation tree in $GK_n^c cut$ and the bottom-most inference is application of rule (cut F). Let it be the only application of the cut rule. Then the derivation tree is of the form:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ S_1 = \Gamma \to \Delta, F\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ S_2 = F, \Pi \to \Lambda\end{array}}{S = \Gamma, \Pi \to \Delta, \Lambda}\ (\text{cut } F)$$

It should be noticed, that $\mathcal{D}_1$ and $\mathcal{D}_2$ do not contain any applications of the cut rule and therefore, they already are derivation trees in GK_n^c. Let's show that this application of (cut F) can be eliminated. The proof is by double induction on the ordered pair $\langle \mathrm{l}(F), h\rangle$, where $h = \mathrm{h}(\mathcal{D}_1) + \mathrm{h}(\mathcal{D}_2)$ and is called the *cut height*. Both $\mathcal{D}_1$ and $\mathcal{D}_2$ contains at least one sequent, therefore the smallest value of the cut height is 2.

1. If $\mathrm{l}(F) = 0$ and $h = 2$, then both S_1 and S_2 are axioms. If F is the main formula of both S_1 and S_2, then S_1 is of the form $F, \Gamma_1 \to \Delta, F$ and S_2 is of the form $F, \Pi \to \Lambda_1, F$. In this case S is of the form $F, \Gamma_1, \Pi \to \Delta, \Lambda_1, F$ and clearly an axiom of GK_n^c. Therefore the application of (cut F) is not needed.

   If F is not the main formula of sequent S_1, then S_1 is of the form $G, \Gamma_1 \to \Delta_1, G, F$ and S is of the form $G, \Gamma_1\Pi \to \Delta_1, G, \Lambda$. Once again it is clear that S is an axiom and the application of $(\text{cut } F)$ can be eliminated. The case when F is not the main formula of S_2 is completely analogous.

2. If $\mathrm{l}(F) = 0$ and $\mathrm{h}(\mathcal{D}_1) > 1$, then all the possible bottom-most inferences of $\mathcal{D}_1$ must be analysed. It is clear that the main formula of such inference is definitely not F.

   (a) In the case of $(\neg \rightarrow)$ rule the derivation tree $\mathcal{D}$ is:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1^{'}\\ \Gamma_1 \rightarrow \Delta, G, F\end{matrix}}{S_1 = \neg G, \Gamma_1 \rightarrow \Delta, F}(\neg \rightarrow) \qquad \begin{matrix}\mathcal{D}_2\\ S_2 = F, \Pi \rightarrow \Lambda\end{matrix}}{S = \neg G, \Gamma_1, \Pi \rightarrow \Delta, \Lambda}(\text{cut } F)$$

   Now let's analyse the following derivation

$$\dfrac{\begin{matrix}\mathcal{D}_1^{'}\\ \Gamma_1 \rightarrow \Delta, G, F\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ S_2 = F, \Pi \rightarrow \Lambda\end{matrix}}{\Gamma_1, \Pi \rightarrow \Delta, G, \Lambda}(\text{cut } F)$$

   The cut height of this proof is smaller than h and the cut formula is the same, therefore according to induction hypothesis, the application of $(\text{cut } F)$ can be eliminated from it to obtain the derivation tree $\mathcal{D}^{'}$. Now the derivation tree of S without cut is:

$$\dfrac{\begin{matrix}\mathcal{D}^{'}\\ \Gamma_1, \Pi \rightarrow \Delta, G, \Lambda\end{matrix}}{\neg G, \Gamma_1, \Pi \rightarrow \Delta, \Lambda}(\neg \rightarrow)$$

   The cases of other logical rules are analogous.

   (b) In the case of $(\rightarrow \Box_l)$ rule derivation tree $\mathcal{D}$ is:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1^{'}\\ \Gamma_2 \rightarrow G\end{matrix}}{S_1 = \Gamma_1, \Box_l \Gamma_2 \rightarrow \Delta_1, \Box_l G, F}(\rightarrow \Box_l) \qquad \begin{matrix}\mathcal{D}_2\\ S_2 = F, \Pi \rightarrow \Lambda\end{matrix}}{S = \Gamma_1, \Box_l \Gamma_2, \Pi \rightarrow \Delta_1, \Box_l G, \Lambda}(\text{cut } F)$$

   Now the application of the cut rule can be eliminated by changing $\mathcal{D}$ to:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1^{'}\\ \Gamma_2 \rightarrow G\end{matrix}}{\Gamma_1, \Box_l \Gamma_2 \rightarrow \Delta_1, \Box_l G}(\rightarrow \Box_l)}{\overline{S = \Gamma_1, \Box_l \Gamma_2, \Pi \rightarrow \Delta_1, \Box_l G, \Lambda}}$$

   The case of rule $(\rightarrow \Box^c)$ is analogous.

3. If $\mathrm{l}(F) = 0$ and $\mathrm{h}(\mathcal{D}_2) > 1$, then all the possible bottom-most inferences of $\mathcal{D}_2$ must be analysed. Once again it is clear that the main formula of such inference is definitely not F. The cases of all the logical rules are analogous to case 2a. In the case of $(\rightarrow \Box_l)$ rule derivation tree $\mathcal{D}$ is:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ S_1 = \Gamma \rightarrow \Delta, F\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2^{'}\\ \Pi_2 \rightarrow G\end{array}}{S_2 = F, \Pi_1, \Box_l \Pi_2 \rightarrow \Lambda_1, \Box_l G}\,(\rightarrow \Box_l)}{S = \Gamma, \Pi_1, \Box_l \Pi_2 \rightarrow \Delta, \Lambda_1, \Box_l G}\,(\text{cut } F)$$

Now the application of the cut rule can be eliminated by changing $\mathcal{D}$ to:

$$\begin{array}{c}\mathcal{D}_2^{'}\\ \dfrac{\Pi_2 \rightarrow G}{\Pi_1, \Box_l \Pi_2 \rightarrow \Lambda_1, \Box_l G}\,(\rightarrow \Box_l)\\ \hline\hline S = \Gamma, \Pi_1, \Box_l \Pi_2 \rightarrow \Delta, \Lambda_1, \Box_l G\end{array}$$

The case of rule $(\rightarrow \Box^c)$ is analogous.

4. If $\mathrm{l}(F) > 0$ and $h = 2$, then this case is analogous to case 1.

5. If $\mathrm{l}(F) > 0$ and $\mathrm{h}(\mathcal{D}_1) > 1$, then all the possible bottom-most inferences of $\mathcal{D}_1$ must be analysed. If F is not the main formula of such inference, then this case is analogous to case 2. Therefore, let F be the main formula of the last inference.

   (a) Let the last inference be application of rule $(\rightarrow \neg)$. Then $\mathcal{D}$ is:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1^{'}\\ G, \Gamma \rightarrow \Delta\end{array}}{S_1 = \Gamma \rightarrow \Delta, \neg G}\,(\rightarrow \neg) \qquad \begin{array}{c}\mathcal{D}_2\\ S_2 = \neg G, \Pi \rightarrow \Lambda\end{array}}{S = \Gamma, \Pi \rightarrow \Delta, \Lambda}\,(\text{cut } \neg G)$$

   Now because the rule $(\rightarrow \neg)$ is invertible in GK_n^c and $\vdash_{GK_n^c} S_2$, sequent $S_2^{'} = \Pi \rightarrow \Lambda, G$ is derivable in GK_n^c too. Let the derivation tree of $S_2^{'}$ be $\mathcal{D}_2^{'}$. Now let's inspect the following derivation:

$$\dfrac{\begin{array}{c}\mathcal{D}_2^{'}\\ S_2^{'} = \Pi \rightarrow \Lambda, G\end{array} \qquad \begin{array}{c}\mathcal{D}_1^{'}\\ G, \Gamma \rightarrow \Delta\end{array}}{S = \Gamma, \Pi \rightarrow \Delta, \Lambda}\,(\text{cut } G)$$

   Because $\mathrm{l}(G) < \mathrm{l}(F)$, according to induction hypothesis the application of the cut rule can be removed from this derivation to get the derivation tree of S in GK_n^c.

(b) Let the last inference be application of rule $(\to\wedge)$. Then $\mathcal{D}$ is:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1^{'} \\ \Gamma \to \Delta, G\end{matrix} \quad \begin{matrix}\mathcal{D}_1^{''} \\ \Gamma \to \Delta, H\end{matrix}}{\Gamma \to \Delta, G \wedge H}(\to\wedge) \qquad \begin{matrix}\mathcal{D}_2 \\ S_2 = G \wedge H, \Pi \to \Lambda\end{matrix}}{S = \Gamma, \Pi \to \Delta, \Lambda}(\text{cut } G \wedge H)$$

Now because the rule $(\to\wedge)$ is invertible in GK_n^c and $\vdash_{GK_n^c} S_2$, sequent $S_2^{'} = G, H, \Pi \to \Lambda$ is derivable in GK_n^c too. Let the derivation tree of $S_2^{'}$ be $\mathcal{D}_2^{'}$. Now let's inspect the following derivation:

$$\dfrac{\begin{matrix}\mathcal{D}_1^{'} \\ \Gamma \to \Delta, G\end{matrix} \qquad \begin{matrix}\mathcal{D}_2^{'} \\ S_2^{'} = G, H, \Pi \to \Lambda\end{matrix}}{H, \Gamma, \Pi \to \Delta, \Lambda}(\text{cut } G)$$

Because $\mathrm{l}(G) < \mathrm{l}(F)$, according to induction hypothesis the application of the cut rule can be removed from this derivation to get the derivation tree $\mathcal{D}_2^{''}$ in GK_n^c. Now consider this derivation:

$$\dfrac{\begin{matrix}\mathcal{D}_1^{''} \\ \Gamma \to \Delta, H\end{matrix} \qquad \begin{matrix}\mathcal{D}_2^{''} \\ H, \Gamma, \Pi \to \Delta, \Lambda\end{matrix}}{\Gamma, \Gamma, \Pi \to \Delta, \Delta, \Lambda}(\text{cut } H)$$

Once again, $\mathrm{l}(H) < \mathrm{l}(F)$, therefore according to induction hypothesis the application of the cut rule can be eliminated to get the derivation tree $\mathcal{D}_3$. Now the derivation tree of S is as follows:

$$\dfrac{\begin{matrix}\mathcal{D}_3 \\ \Gamma, \Gamma, \Pi \to \Delta, \Delta, \Lambda\end{matrix}}{\Gamma, \Pi \to \Delta, \Lambda}$$

The cases, when the last inference is application of rules $(\vee\to)$ or $(\supset\to)$ are analogous to case 5a.

(c) Let the last inference be application of rule $(\to\Box_l)$. In this case all the possible variants of $\mathcal{D}_2$ must be considered.

i. If $\mathrm{h}(\mathcal{D}_2) = 1$, then S_2 is an axiom. If F is not the main formula of axiom, then S_2 is of the form $F, G, \Pi_1 \to \Lambda_1, G$ and S is of the form $\Gamma, G, \Pi_1 \to \Delta\Lambda_1, G$. It is clear that S is an axiom of GK_n^c. If F is the main formula of axiom S_2, then $\mathcal{D}$ is:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1^{'} \\ \Gamma_2 \to G\end{matrix}}{S_1 = \Gamma_1, \Box_l\Gamma_2 \to \Delta, \Box_l G}(\to\Box_l) \qquad \begin{matrix}\mathcal{D}_2 \\ S_2 = \Box_l G, \Pi \to \Lambda_1, \Box_l G\end{matrix}}{S = \Gamma_1, \Box_l\Gamma_2, \Pi \to \Delta, \Lambda_1, \Box_l G}(\text{cut } \Box_l G)$$

Now the application of the cut rule can be eliminated by changing $\mathcal{D}$ to:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Gamma_2 \to G\end{matrix}}{\Gamma_1, \Box_l\Gamma_2 \to \Delta, \Box_l G}\,(\to \Box_l)}{S = \Gamma_1, \Box_l\Gamma_2, \Pi \to \Delta, \Lambda_1, \Box_l G}$$

ii. If $\mathrm{h}(\mathcal{D}_2) > 1$ and the last inference in $\mathcal{D}_2$ is application of logical rule or rule $(\to \Box_{l_1})$, where $l_1 \neq l$, then the cut rule is eliminated from $\mathcal{D}$ analogously to case 3.

iii. If $\mathrm{h}(\mathcal{D}_2) > 1$ and the last inference in $\mathcal{D}_2$ is application of rule $(\to \Box_l)$, then $\mathcal{D}$ is:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Gamma_2 \to G\end{matrix}}{S_1 = \Gamma_1, \Box_l\Gamma_2 \to \Delta, \Box_l G}\,(\to \Box_l) \qquad \dfrac{\begin{matrix}\mathcal{D}_2'\\ G, \Pi_2 \to H\end{matrix}}{S_2 = \Box_l G, \Pi_1, \Box_l\Pi_2 \to \Lambda_1, \Box_l H}\,(\to \Box_l)}{S = \Gamma_1, \Box_l\Gamma_2, \Pi_1, \Box_l\Pi_2 \to \Delta, \Lambda_1, \Box_l H}\,(\text{cut } \Box_l G)$$

Now consider the following derivation:

$$\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Gamma_2 \to G\end{matrix} \qquad \begin{matrix}\mathcal{D}_2'\\ G, \Pi_2 \to H\end{matrix}}{\Gamma_2, \Pi_2 \to H}\,(\text{cut } G)$$

According to induction hypothesis, because $\mathrm{l}(G) < \mathrm{l}(F)$ it is possible to eliminate cut from this derivation to get the derivation tree $\mathcal{D}_3$. Then the derivation tree of S without cut is:

$$\dfrac{\begin{matrix}\mathcal{D}_3\\ \Gamma_2, \Pi_2 \to H\end{matrix}}{S = \Gamma_1, \Box_l\Gamma_2, \Pi_1, \Box_l\Pi_2 \to \Delta, \Lambda_1, \Box_l H}\,(\to \Box_l)$$

iv. If $\mathrm{h}(\mathcal{D}_2) > 1$ and the last inference in $\mathcal{D}_2$ is application of rule $(\to \Box^c)$, then $\mathcal{D}$ is:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Gamma_2 \to G\end{matrix}}{S_1 = \Gamma_1, \Box_l\Gamma_2 \to \Delta, \Box_l G}\,(\to \Box_l) \qquad \dfrac{\begin{matrix}\mathcal{D}_2'\\ G, \Pi_2 \to H\end{matrix}}{S_2 = \Box_l G, \Pi_1, \Box_*\Pi_2 \to \Lambda_1, \Box_c H}\,(\to \Box^c)}{S = \Gamma_1, \Box_l\Gamma_2, \Pi_1, \Box_*\Pi_2 \to \Delta, \Lambda_1, \Box_c H}\,(\text{cut } \Box_l G)$$

Now consider the following derivation:

$$\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Gamma_2 \to G\end{matrix} \qquad \begin{matrix}\mathcal{D}_2'\\ G, \Pi_2 \to H\end{matrix}}{\Gamma_2, \Pi_2 \to H}\,(\text{cut } G)$$

According to induction hypothesis, because $\mathrm{l}(G) < \mathrm{l}(F)$ it is possible to eliminate cut from this derivation to get the derivation tree $\mathcal{D}_3$. Then the derivation tree of S without cut is:

$$\dfrac{\begin{array}{c}\mathcal{D}_3\\ \Gamma_2, \Pi_2 \to H\end{array}}{S = \Gamma_1, \Box_l\Gamma_2, \Pi_1, \Box_*\Pi_2 \to \Delta, \Lambda_1, \Box_c H}\ (\to \Box^c)$$

(d) Let the last inference be application of rule $(\to \Box^c)$. Once again all the possible variants of $\mathcal{D}_2$ must be considered.

i. If $\mathrm{h}(\mathcal{D}_2) = 1$, then this case is analogous to case 5(c)i.

ii. If $\mathrm{h}(\mathcal{D}_2) > 1$ and the last inference in $\mathcal{D}_2$ is application of logical rule or rule $(\to \Box_a)$, where $a \neq c$, then this case is analogous to case 5(c)ii.

iii. If $\mathrm{h}(\mathcal{D}_2) > 1$ and the last inference in $\mathcal{D}_2$ is application of rule $(\to \Box_c)$, then $\mathcal{D}$ is:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma_2 \to G\end{array}}{S_1 = \Gamma_1, \Box_*\Gamma_2 \to \Delta, \Box_c G}\ (\to \Box^c) \qquad \dfrac{\begin{array}{c}\mathcal{D}_2'\\ G, \Pi_2 \to H\end{array}}{S_2 = \Box_c G, \Pi_1, \Box_c\Pi_2 \to \Lambda_1, \Box_c H}\ (\to \Box_c)}{S = \Gamma_1, \Box_*\Gamma_2, \Pi_1, \Box_c\Pi_2 \to \Delta, \Lambda_1, \Box_c H}\ (\text{cut } \Box_c G)$$

Now consider the following derivation:

$$\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma_2 \to G\end{array} \qquad \begin{array}{c}\mathcal{D}_2'\\ G, \Pi_2 \to H\end{array}}{\Gamma_2, \Pi_2 \to H}\ (\text{cut } G)$$

According to induction hypothesis, because $\mathrm{l}(G) < \mathrm{l}(F)$ it is possible to eliminate cut from this derivation to get the derivation tree $\mathcal{D}_3$. Then the derivation tree of S without cut is:

$$\dfrac{\begin{array}{c}\mathcal{D}_3\\ \Gamma_2, \Pi_2 \to H\end{array}}{S = \Gamma_1, \Box_*\Gamma_2, \Pi_1, \Box_c\Pi_2 \to \Delta, \Lambda_1, \Box_c H}\ (\to \Box^c)$$

iv. If $\mathrm{h}(\mathcal{D}_2) > 1$ and the last inference in $\mathcal{D}_2$ is application of rule $(\to \Box^c)$, then this case is analogous to case 5(c)iv.

6. If $\mathrm{l}(F) > 0$ and $\mathrm{h}(\mathcal{D}_2) > 1$, then all the possible bottom-most inferences of $\mathcal{D}_2$ must be analysed.

(a) If the last inference is application of logical rule and F is the main formula, then this case is analogous to case 5a or 5b.

(b) If (1) the last inference is application of logical rule and F is not the main formula, (2) the last inference is application of rule $(\to \Box_l)$ and F is not of the form $\Box_l G$ or (3) the last inference is application of rule $(\rightarrow \Box^c)$ and F is not of the form $\Box_l G$, where l is any agent, then this case is analogous to case 3.

(c) Let the last inference be application of rule $(\rightarrow \Box_l)$ and F be of the form $\Box_l G$. In this case all the possible variants of $\mathcal{D}_1$ must be considered.

i. If $\mathrm{h}(\mathcal{D}_1) = 1$, then this case is analogous to case 5(c)i.

ii. If $\mathrm{h}(\mathcal{D}_1) > 1$ and the last inference in $\mathcal{D}_1$ is application of logical rule, rule $(\rightarrow \Box_{l_1})$ ($l_1 = l$ or $l_1 \neq l$) or rule $(\rightarrow \Box^c)$ and F is not the main formula of the application, then the cut rule is eliminated from $\mathcal{D}$ analogously to case 2.

iii. If $\mathrm{h}(\mathcal{D}_1) > 1$, the last inference in $\mathcal{D}_1$ is application of rule $(\rightarrow \Box_l)$ and the main formula of the application is F, then this case is already covered in case 5(c)iii.

iv. If $\mathrm{h}(\mathcal{D}_1) > 1$, the last inference in $\mathcal{D}_1$ is application of rule $(\rightarrow \Box^c)$, $l = c$ and the main formula of the application is F, then this case is already covered in case 5(d)iii.

(d) Let the last inference be application of rule $(\rightarrow \Box^c)$. Once again all the possible variants of $\mathcal{D}_1$ must be considered. However all the cases are either already covered in the proof or proved analogously to the ones already presented.

$\Box$

Theorem 2.4.7 (Cut elimination for $K4_n^c$)**.** *Sequent is derivable in $GK4_n^c cut$ iff it is derivable in $GK4_n^c$.*

Proof. The proof is analogous to the proof of Theorem 2.4.7. Once again, the aim of the prove is to show that the application of the cut rule can be eliminated from any derivation tree, that contains only one application of the cut rule and it is the last inference in the derivation tree. Let $\mathcal{D}$ be such derivation tree in $GK4_n^c cut$:

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ S_1 = \Gamma \rightarrow \Delta, F\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ S_2 = F, \Pi \rightarrow \Lambda\end{matrix}}{S = \Gamma, \Pi \rightarrow \Delta, \Lambda} \text{ (cut } F\text{)}$$

For the proof double induction on the ordered pair $\langle \mathrm{l}(F), h \rangle$, where $h = \mathrm{h}(\mathcal{D}_1) + \mathrm{h}(\mathcal{D}_2)$ is used. The only difference between GK_n^c and $GK4_n^c$ is modal rules $(\rightarrow \Box_l)$ and $(\rightarrow \Box^c)$, therefore only the cases, that involve those rules are discussed.

Case 2b, where $l(F) = 0$, $h(\mathcal{D}_1) > 1$ and rule $(\to \Box_l)$ is applied to S_1. Derivation tree $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Gamma_2, \Box_l\Gamma_2 \to G\end{matrix}}{S_1 = \Gamma_1, \Box_l\Gamma_2 \to \Delta_1, \Box_l G, F}\,(\to \Box_l) \qquad \begin{matrix}\mathcal{D}_2\\ S_2 = F, \Pi \to \Lambda\end{matrix}}{S = \Gamma_1, \Box_l\Gamma_2, \Pi \to \Delta_1, \Box_l G, \Lambda}\,(\text{cut } F)$$

Now the application of the cut rule can be eliminated by changing $\mathcal{D}$ to:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Gamma_2, \Box_l\Gamma_2 \to G\end{matrix}}{\Gamma_1, \Box_l\Gamma_2 \to \Delta_1, \Box_l G}\,(\to \Box_l)}{\overline{S = \Gamma_1, \Box_l\Gamma_2, \Pi \to \Delta_1, \Box_l G, \Lambda}}$$

The case of rule $(\to \Box^c)$ is analogous.

Case 3, where $l(F) = 0$, $h(\mathcal{D}_2) > 1$. Only the part, when rule $(\to \Box_l)$ or $(\to \Box^c)$ is applied to S_2 is different. However the changes are obvious and similar to the previous case.

Cases 5(c)iii and 6(c)iii, where $l(F) = 0$, $h(\mathcal{D}_1) > 1$, $h(\mathcal{D}_2) > 1$, rule $(\to \Box_l)$ is applied to S_1 and F is the main formula of the application and $(\to \Box_l)$ is applied to S_2. Then $\mathcal{D}$ looks like:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Gamma_2, \Box_l\Gamma_2 \to G\end{matrix}}{S_1 = \Gamma_1, \Box_l\Gamma_2 \to \Delta, \Box_l G}\,(\to \Box_l) \qquad \dfrac{\begin{matrix}\mathcal{D}_2\\ G, \Box_l G, \Pi_2, \Box_l\Pi_2 \to H\end{matrix}}{S_2 = \Box_l G, \Pi_1, \Box_l\Pi_2 \to \Lambda_1, \Box_l H}\,(\to \Box_l)}{S = \Gamma_1, \Box_l\Gamma_2, \Pi_1, \Box_l\Pi_2 \to \Delta, \Lambda_1, \Box_l H}\,(\text{cut } \Box_l G)$$

Now consider the following derivation:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Gamma_2, \Box_l\Gamma_2 \to G\end{matrix}}{S_1 = \Box_l\Gamma_2 \to \Box_l G}\,(\to \Box_l) \qquad \begin{matrix}\mathcal{D}_2'\\ G, \Box_l G, \Pi_2, \Box_l\Pi_2 \to H\end{matrix}}{\Box_l\Gamma_2, G, \Pi_2, \Box_l\Pi_2 \to H}\,(\text{cut } \Box_l G)$$

Now the cut height of this derivation is smaller than the cut height of $\mathcal{D}$ and the cut formula is the same, therefore according to induction hypothesis cut rule can be eliminated from this derivation to get derivation tree $\mathcal{D}_2''$ without cut. Next let's analyse this derivation:

$$\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Gamma_2, \Box_l\Gamma_2 \to G\end{matrix} \qquad \begin{matrix}\mathcal{D}_2'\\ \Box_l\Gamma_2, G, \Pi_2, \Box_l\Pi_2 \to H\end{matrix}}{\Gamma_2, \Box_l\Gamma_2, \Box_l\Gamma_2\Pi_2, \Box_l\Pi_2 \to H}\,(\text{cut } G)$$

According to induction hypothesis, because $\mathrm{l}(G) < \mathrm{l}(F)$ it is possible to eliminate cut from this derivation to get the derivation tree $\mathcal{D}_3$. Then the derivation tree of S without cut is:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_3}{\Gamma_2, \Box_l\Gamma_2, \Box_l\Gamma_2\Pi_2, \Box_l\Pi_2 \to H}}{\Gamma_2, \Box_l\Gamma_2, \Pi_2, \Box_l\Pi_2 \to H}}{S = \Gamma_1, \Box_l\Gamma_2, \Pi_1, \Box_l\Pi_2 \to \Delta, \Lambda_1, \Box_l H}\ (\to \Box_l)$$

Cases 5(c)iv, 5(d)iii and 5(d)iv are changed in the same way as the previous one and they already cover case 6(c)iv and part of case 6d that needs alteration. □

Theorem 2.4.8 (Cut elimination for T_n^c)**.** *Sequent is derivable in $GT_n^c cut$ iff it is derivable in GT_n^c.*

Proof. The proof is analogous to the proof of Theorem 2.4.6. Once again, the same notation and the same double induction is used. The only difference between GK_n^c and GT_n^c is that the latter contains modal rule $(\Box_l \to)$, which is not part of the former. Therefore only the new cases, that involve this rule, are discussed.

1. In case 2, where $\mathrm{l}(F) = 0$ and $\mathrm{h}(\mathcal{D}_1) > 1$, one more variant must be analysed, in which rule $(\Box_l \to)$ is applied to S_1, however this case is analogous to case 2a of the proof of Lemma 2.4.6.

2. In case 3, where $\mathrm{l}(F) = 0$ and $\mathrm{h}(\mathcal{D}_2) > 1$, one more variant must be analysed, in which rule $(\Box_l \to)$ is applied to S_2, however this case is also analogous to case 2a of the proof of Lemma 2.4.6.

3. In case 5c, where $\mathrm{l}(F) > 0$, $\mathrm{h}(\mathcal{D}_1) > 1$ and rule $(\to \Box_l)$ is applied to S_1, two more variants must be analysed. First of all, if rule $(\Box_{l_1} \to)$ (where $l_1 = l$ or $l_1 \neq l$) is applied to S_2 and F is not the main formula of the application, then this case is analogous to the case 5(c)ii of the proof of Lemma 2.4.6.

   If however F is the main formula of the inference, then $\mathcal{D}$ is:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1'}{\Gamma_2 \to G}}{S_1 = \Gamma_1, \Box_l\Gamma_2 \to \Delta, \Box_l G}\ (\to \Box_l) \qquad \dfrac{\dfrac{\mathcal{D}_2'}{G, \Box_l G, \Pi \to \Lambda}}{S_2 = \Box_l G, \Pi \to \Lambda}\ (\Box_l \to)}{S = \Gamma_1, \Box_l\Gamma_2, \Pi \to \Delta, \Lambda}\ (\mathrm{cut}\ \Box_l G)$$

Now consider the following derivation:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1' \\ \Gamma_2 \to G\end{matrix}}{S_1 = \Gamma_1, \Box_l\Gamma_2 \to \Delta, \Box_l G}\,(\to \Box_l) \qquad \begin{matrix}\mathcal{D}_2' \\ G, \Box_l G, \Pi \to \Lambda\end{matrix}}{\Gamma_1, \Box_l\Gamma_2, G, \Pi \to \Delta, \Lambda}\,(\text{cut } \Box_l G)$$

The cut height of this derivation is smaller than that of $\mathcal{D}$ and the cut formula is the same, therefore it is possible to remove this cut to get the derivation tree $\mathcal{D}_2''$. Now let's analyse the following derivation:

$$\dfrac{\begin{matrix}\mathcal{D}_1' \\ \Gamma_2 \to G\end{matrix} \qquad \begin{matrix}\mathcal{D}_2'' \\ \Gamma_1, \Box_l\Gamma_2, G, \Pi \to \Delta, \Lambda\end{matrix}}{\Gamma_1, \Gamma_2, \Box_l\Gamma_2, \Pi \to \Delta, \Lambda}\,(\text{cut } G)$$

According to induction hypothesis, because $\mathrm{l}(G) < \mathrm{l}(F)$ it is possible to eliminate cut from this derivation to get the derivation tree $\mathcal{D}_3$. Then the derivation tree of S without cut is:

$$\dfrac{\begin{matrix}\mathcal{D}_3 \\ \Gamma_1, \Gamma_2, \Box_l\Gamma_2, \Pi \to \Delta, \Lambda\end{matrix}}{S = \Gamma_1, \Box_l\Gamma_2, \Pi \to \Delta, \Lambda}\,(\Box_l \to),\ldots,(\Box_l \to)$$

4. In case 5d, where $\mathrm{l}(F) > 0$, $\mathrm{h}(\mathcal{D}_1) > 1$ and rule $(\to \Box^c)$ is applied to S_1, two more variants must be analysed. The alteration is completely analogous to the previous case.

5. In case 6, where $\mathrm{l}(F) > 0$ and $\mathrm{h}(\mathcal{D}_2) > 1$, one more variant must be added, in which rule $(\Box_l \to)$ is applied to S_2. If F is not the main formula of the application, then this case is analogous to case 3 of the proof of Lemma 2.4.6.

   However if F is the main formula of the application, then all the possible cases of derivation tree $\mathcal{D}_1$ must be checked.

   (a) If $\mathrm{h}(\mathcal{D}_1) = 1$, then S_1 is an axiom. If F is not the main formula of the axiom, then S_1 is of the form $G, \Gamma_1 \to \Delta_1, G, F$ and S is of the form $G, \Gamma_1, \Pi \to \Delta_1, G, \Lambda$. It is clear that S is an axiom of GT_n^c. If F is the main formula of axiom S_1, then $\mathcal{D}$ is:

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ S_1 = \Box_l G, \Gamma_1 \to \Delta, \Box_l G\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_2' \\ G, \Box_l G, \Pi \to \Lambda\end{matrix}}{S_2 = \Box_l G, \Pi \to \Lambda}\,(\Box_l \to)}{S = \Box_l G, \Gamma_1, \Pi \to \Delta, \Lambda}\,(\text{cut } \Box_l G)$$

Now the application of the cut rule can be eliminated by changing $\mathcal{D}$ to:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_2^{'} \\ G, \Box_l G, \Pi \rightarrow \Lambda\end{matrix}}{S_2 = \Box_l G, \Pi \rightarrow \Lambda}\,(\Box_l \rightarrow)}{S = \Box_l G, \Gamma_1, \Pi \rightarrow \Delta, \Lambda}$$

(b) If $\mathrm{h}(\mathcal{D}_1) > 1$ and the last inference in $\mathcal{D}_1$ is application of logical rule, rule $(\Box_{l_1} \rightarrow)$ (where $l_1 = l$ or $l_1 \neq l$), rule $(\rightarrow \Box_{l_2})$ (where $l_2 = l$ or $l_2 \neq l$) or rule $(\rightarrow \Box^c)$ and F is not the main formula of the application, then the cut rule is eliminated from $\mathcal{D}$ analogously to case 2 of the proof of Lemma 2.4.6.

(c) If $\mathrm{h}(\mathcal{D}_1) > 1$, the last inference in $\mathcal{D}_1$ is application of rule $(\rightarrow \Box_l)$ and F is the main formula of the application, then this situation is already covered in case 3 of this proof.

(d) If $\mathrm{h}(\mathcal{D}_1) > 1$, the last inference in $\mathcal{D}_1$ is application of rule $(\rightarrow \Box^c)$ and F is the main formula of the application, then this situation is also covered in case 4 of this proof.

6. In cases 6c and 6d, where $\mathrm{l}(F) > 0$, $\mathrm{h}(\mathcal{D}_2) > 1$, and respectively rule $(\rightarrow \Box_l)$ or $(\rightarrow \Box^c)$ is applied to S_2, one more variant must be analysed, in which rule $(\Box_l \rightarrow)$ is applied to S_1. However these cases are analogous to case 2a of the proof of Lemma 2.4.6.

$\Box$

Theorem 2.4.9 (Cut elimination for $S4^c_n$)**.** *Sequent is derivable in $GS4^c_n cut$ iff it is derivable in $GS4^c_n$.*

Proof. This proof is also analogous to the proof of Theorem 2.4.6. However $GS4^c_n$ is closer to GT^c_n than to GK^c_n. Calculus $GS4^c_n$ includes rule $(\Box_l \rightarrow)$, which is also part of GT^c_n, but not of GK^c_n. Nevertheless, rules $(\rightarrow \Box_l)$ and $(\rightarrow \Box^c)$ are different in $GS4^c_n$, than in both calculi GK^c_n and GT^c_n. Therefore alterations made in the proof of Lemma 2.4.8 (referred to as proof for T^c_n) must also be part of this proof. However, due to different $(\rightarrow \Box_l)$ and $(\rightarrow \Box^c)$ rules, new changes must be incorporated both in the proof for T^c_n and in the proof of Lemma 2.4.6 (referred to as proof for K^c_n).

In case 2b of the proof for K_n^c, where $\mathrm{l}(F) = 0$, $\mathrm{h}(\mathcal{D}_1) > 1$ and rule $(\to \Box_l)$ is applied to S_1, derivation tree $\mathcal{D}$ looks like:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Box_l\Gamma_2 \to G\end{matrix}}{S_1 = \Gamma_1, \Box_l\Gamma_2 \to \Delta_1, \Box_l G, F}\,(\to \Box_l) \qquad \begin{matrix}\mathcal{D}_2\\ S_2 = F, \Pi \to \Lambda\end{matrix}}{S = \Gamma_1, \Box_l\Gamma_2, \Pi \to \Delta_1, \Box_l G, \Lambda}\,(\mathrm{cut}\ F)$$

Now the application of the cut rule can be eliminated by changing $\mathcal{D}$ to:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Box_l\Gamma_2 \to G\end{matrix}}{\Gamma_1, \Box_l\Gamma_2 \to \Delta_1, \Box_l G}\,(\to \Box_l)}{S = \Gamma_1, \Box_l\Gamma_2, \Pi \to \Delta_1, \Box_l G, \Lambda}$$

The case of rule $(\to \Box^c)$ is analogous.

In case 3 of the proof for K_n^c, where $\mathrm{l}(F) = 0$ and $\mathrm{h}(\mathcal{D}_2) > 1$, only the part, when rule $(\to \Box_l)$ or $(\to \Box^c)$ is applied to S_2 is different. However the changes are obvious and similar to the previous case of this proof.

In case 5c of the proof for K_n^c and case 3 of the proof for T_n^c, where $\mathrm{l}(F) = 0$, $\mathrm{h}(\mathcal{D}_2) > 1$, rule $(\to \Box_l)$ is applied to S_1 and F is the main formula of the application, all the possible variants of $\mathcal{D}_2$ must be altered.

1. If $\mathrm{h}(\mathcal{D}_2) = 1$, then the change is obvious.
2. If $\mathrm{h}(\mathcal{D}_2) > 1$ and the last inference in $\mathcal{D}_2$ is application of logical rule, rule $(\to \Box_{l_1})$, where $l_1 \neq l$, or rule $(\Box_{l_2} \to)$ (where $l_2 = l$ or $l_2 \neq l$) and F is not the main formula of the application, then the cut rule is eliminated from $\mathcal{D}$ analogously to case 3 of the proof for K_n^c.
3. If $\mathrm{h}(\mathcal{D}_2) > 1$ and the last inference in $\mathcal{D}_2$ is application of rule $(\to \Box_l)$, then $\mathcal{D}$ is:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Box_l\Gamma_2 \to G\end{matrix}}{S_1 = \Gamma_1, \Box_l\Gamma_2 \to \Delta, \Box_l G}\,(\to \Box_l) \qquad \dfrac{\begin{matrix}\mathcal{D}_2'\\ \Box_l G, \Box_l\Pi_2 \to H\end{matrix}}{S_2 = \Box_l G, \Pi_1, \Box_l\Pi_2 \to \Lambda_1, \Box_l H}\,(\to \Box_l)}{S = \Gamma_1, \Box_l\Gamma_2, \Pi_1, \Box_l\Pi_2 \to \Delta, \Lambda_1, \Box_l H}\,(\mathrm{cut}\ \Box_l G)$$

Now consider the following derivation:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Box_l\Gamma_2 \to G\end{matrix}}{S_1 = \Box_l\Gamma_2 \to \Box_l G}\,(\to \Box_l) \qquad \begin{matrix}\mathcal{D}_2'\\ \Box_l G, \Box_l\Pi_2 \to H\end{matrix}}{\Box_l\Gamma_2, \Box_l\Pi_2 \to H}\,(\mathrm{cut}\ \Box_l G)$$

The cut height of this derivation is smaller that that of derivation tree $\mathcal{D}$ and the cut formula is the same, therefore according to induction hypothesis, it is possible to eliminate cut from this derivation to get the derivation tree $\mathcal{D}_3$. Then the derivation tree of S without cut is:

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}_3\\ \Box_l\Gamma_2, \Box_l\Pi_2 \to H\end{array}}{\Box_l\Gamma_2, \Box_l\Pi_2 \to \Box_l H}\,(\to \Box_l)}{S = \Gamma_1, \Box_l\Gamma_2, \Pi_1, \Box_l\Pi_2 \to \Delta, \Lambda_1, \Box_l H}$$

4. If $\mathrm{h}(\mathcal{D}_2) > 1$ and the last inference in $\mathcal{D}_2$ is application of rule $(\to \Box^c)$, then the change is analogous to the previous case of this proof.

5. If $\mathrm{h}(\mathcal{D}_2) > 1$, the last inference in $\mathcal{D}_2$ is application of rule $(\Box_l \to)$ and F is the main formula of the application, then $\mathcal{D}$ is:

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}_1^{'}\\ \Box_l\Gamma_2 \to G\end{array}}{S_1 = \Gamma_1, \Box_l\Gamma_2 \to \Delta, \Box_l G}\,(\to \Box_l) \quad \cfrac{\begin{array}{c}\mathcal{D}_2^{'}\\ G, \Box_l G, \Pi \to \Lambda\end{array}}{S_2 = \Box_l G, \Pi \to \Lambda}\,(\Box_l \to)}{S = \Gamma_1, \Box_l\Gamma_2, \Pi \to \Delta, \Lambda}\,(\text{cut } \Box_l G)$$

Now consider the following derivation:

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}_1^{'}\\ \Box_l\Gamma_2 \to G\end{array}}{S_1 = \Gamma_1, \Box_l\Gamma_2 \to \Delta, \Box_l G}\,(\to \Box_l) \quad \begin{array}{c}\mathcal{D}_2^{'}\\ G, \Box_l G, \Pi \to \Lambda\end{array}}{\Gamma_1, \Box_l\Gamma_2, G, \Pi \to \Delta, \Lambda}\,(\text{cut } \Box_l G)$$

The cut height of this derivation is smaller than that of $\mathcal{D}$ and the cut formula is the same, therefore it is possible to remove this cut to get the derivation tree $\mathcal{D}_2^{''}$. Now let's analyse the following derivation tree:

$$\cfrac{\begin{array}{c}\mathcal{D}_1^{'}\\ \Box_l\Gamma_2 \to G\end{array} \quad \begin{array}{c}\mathcal{D}_2^{''}\\ \Gamma_1, \Box_l\Gamma_2, G, \Pi \to \Delta, \Lambda\end{array}}{\Gamma_1, \Box_l\Gamma_2, \Box_l\Gamma_2, \Pi \to \Delta, \Lambda}\,(\text{cut } G)$$

According to induction hypothesis, because $\mathrm{l}(G) < \mathrm{l}(F)$ it is possible to eliminate cut from this derivation to get the derivation tree $\mathcal{D}_3$. Then the derivation tree of S without cut is:

$$\cfrac{\begin{array}{c}\mathcal{D}_3\\ \Gamma_1, \Box_l\Gamma_2, \Box_l\Gamma_2, \Pi \to \Delta, \Lambda\end{array}}{S = \Gamma_1, \Box_l\Gamma_2, \Pi \to \Delta, \Lambda}$$

In case 5d of the proof for K_n^c and case 4 of the proof for T_n^c, where $\mathrm{l}(F) = 0$, $\mathrm{h}(\mathcal{D}_2) > 1$, rule $(\rightarrow \Box^c)$ is applied to S_1 and F is the main formula of the application, the changes are analogous to the previous case of this proof.

The other cases, that must be altered, are either already covered in this proof, or are changed in a similar way, as the ones presented here. Once again, the proof for K_n^c and the proof for T_n^c lists those cases in more detail, therefore there is no need to repeat them again. $\Box$

The proofs of cut elimination theorems demonstrate that calculi GK_n^c, $GK4_n^c$, GT_n^c and $GS4_n^c$ are complete. The soundness of the calculi is obvious, because if some sequent is derivable in a calculus without cut, then it is derivable with the same derivation in respective calculus with cut. There is no necessity to use the cut rule in the derivation search.

Chapter 3

Terminating Calculi for Multimodal Logics with Interaction

In order to automate derivation search in Gentzen-type calculi, two conditions must be met. First of all, it must be clear, what to do in every step. A process of bottom-up derivation search, detailed in Definition 1.4.9, answers this question quite well. Nevertheless, one more condition must be satisfied: in every rule it must be obvious how to obtain the premise from the conclusion. The rule, which has this property, is said to be *analytical*. For example in the cut rule it is not clear how to choose a cut formula. There is an infinite set of possible cut formulas to try, therefore the cut rule is not analytical and the calculi containing it are not analytical. That is why it was so important to remove the cut in Section 2.4. It is not hard to see that calculi GK_n^c, $GK4_n^c$, GT_n^c and $GS4_n^c$ are analytical.

Another property of the calculus, which is important for the automation, is finite derivation search. Indeed every computer program must terminate and provide an answer if the sequent is derivable, or not. However, among the mentioned analytical calculi only GK_n^c is terminating (as will be proved later). The aim of this chapter is to derive terminating calculi for logics in question.

3.1 Logic K_n^c

As mentioned earlier, GK_n^c is terminating. This section is dedicated to proving that.

Lemma 3.1.1. *If S_1 and S_2 are part of derivation search tree in calculus GK_n^c and for some inference S_1 is a conclusion and S_2 is some premise, then* $\mathrm{l}(S_2) < \mathrm{l}(S_1)$.

Proof. If the inference is application of some logical rule, then S_2 consists of the same formulas as S_1, except that the main formula is part of S_1 only, and the side formula(s) is (are) part of S_2 only. Let the main formula be F_1. If there is one side formula F_2, then $\mathrm{l}(F_1) = \mathrm{l}(F_2) + 1$. If there are two side formulas $F_2^{'}$ and $F_2^{''}$, then $\mathrm{l}(F_1) = \mathrm{l}(F_2^{'}) + \mathrm{l}(F_2^{''}) + 1$. In both cases $\mathrm{l}(S_2) < \mathrm{l}(S_1)$.

Let the inference be application of $(\to \Box_l)$ rule:

$$\frac{S_2 = \Gamma_2 \to F}{S_1 = \Gamma_1, \Box_l\Gamma_2 \to \Delta, \Box_l F} \, (\to \Box_l)$$

Now once again $\mathrm{l}(\Box_l F) = \mathrm{l}(F)+1$. Moreover, $\mathrm{l}(\Gamma_2) \leqslant \mathrm{l}(\Box_l\Gamma_2)$ (the situation, when the two lengths are equal, occur if $\Gamma_2 = \varnothing$). Finally, Γ_1 and Δ are not even part of S_2. From this it follows that $\mathrm{l}(S_2) < \mathrm{l}(S_1)$. The case of rule $(\to \Box^c)$ is analogous. $\Box$

Lemma 3.1.2. *For any sequent S if* $\mathrm{l}(S) = 0$, *then no rule of calculus GK_n^c can be applied to S.*

Proof. It is obvious, that every main formula of every rule in GK_n^c must have at least one logical operator. Therefore length of S must be at least 1 and therefore no rule can be applied, if $\mathrm{l}(S) = 0$. $\Box$

Theorem 3.1.3. *Calculus GK_n^c is terminating.*

Proof. Lemmas 3.1.1 and 3.1.2 show that any derivation search tree in GK_n^c terminates. So it is not possible to construct an infinite derivation search tree. Because of that and because to any sequent only finite number of rules can be applied the number of different derivation search trees of one sequent is also finite. So according to Definition 1.4.14 it must be concluded that GK_n^c is terminating. $\Box$

3.2 Logic T_n^c

Calculus GT_n^c is not terminating. Derivation search trees in it may contain loops. It is said, that sequent S_2 subsumes S_1 (denoted $S_2 \succcurlyeq S_1$), if S_2 can be obtained from S_1 by backward-applying the contraction structural rules. A *loop* is a path of some derivation search tree $\mathcal{D}$ from S_1 to S_2, where S_2 is higher in $\mathcal{D}$ and $S_2 \succcurlyeq S_1$. That is S_1 and S_2 consist of the same formulas. However it can not always be said that such sequents are equal, because the antecedent and succedent are multisets and the number of the same formula in S_2 can be larger than in S_1. If a derivation search tree contains one loop, then it is possible to construct another one and that means that it is possible to get an infinite derivation search tree. Moreover, loops can be part of derivation trees too.

The main cause of loops in T_n^c is rule $(\Box_l \to)$. It is not hard to see, that to form an infinite derivation search tree it is possible to apply this rule to the same formula again and again:

$$\cfrac{\cfrac{\cfrac{\ldots}{F, F, \Box_l F, \Gamma \to \Delta}\,(\Box_l \to)}{F, \Box_l F, \Gamma \to \Delta}\,(\Box_l \to)}{\Box_l F, \Gamma \to \Delta}\,(\Box_l \to)$$

However it is not hard to limit such applications by simply labelling the formula, which was the main formula of application of reflexivity rule. As offered in [45] for logic *S4*, let's just mark the outermost $\Box_l$ operator of such formula by star: $\Box_l^*$. Therefore, the reflexivity rule must be changed. What is more, the definition of $(\to \Box_l)$ and $(\to \Box^c)$ rules must be altered to deal with stars in sequents.

Definition 3.2.1. *Terminating Gentzen-type calculus for multimodal logic* T_n^c *(*$G^*T_n^c$*) contains the same axiom and rules as calculus* GT_n^c*, except rules* $(\Box_l \to)$, $(\to \Box_l)$ *and* $(\to \Box^c)$ *are changed to:*

$$\frac{F, \Box_l^* F, \Gamma \to \Delta}{\Box_l F, \Gamma \to \Delta}\,(\Box_l \to)^*$$

$$\frac{\Gamma_2, \Gamma_3 \to F}{\Gamma_1, \Box_l \Gamma_2, \Box_l^* \Gamma_3 \to \Delta, \Box_l F}\,(\to \Box_l)^* \qquad \frac{\Gamma_2, \Gamma_3 \to F}{\Gamma_1, \Box_* \Gamma_2, \Box_*^* \Gamma_3 \to \Delta, \Box_c F}\,(\to \Box^c)^*$$

Theorem 3.2.2. *Calculus* $G^*T_n^c$ *is sound.*

Proof. If sequent S is derivable in $G^*T_n^c$, then by removing all the stars and by changing all the applications of $(\Box_l \to)^*$, $(\to \Box_l)^*$ and $(\to \Box^c)^*$ rules to applications of $(\Box_l \to)$, $(\to \Box_l)$ and $(\to \Box^c)$ rules respectively it is possible to obtain derivation tree in GT_n^c. From the soundness of GT_n^c it follows that $G^*T_n^c$ is also sound. $\Box$

To prove the completeness let's define one intermediate calculus.

Definition 3.2.3. *Let $G_1T_n^c$ be calculus obtained from $G^*T_n^c$ by adding the following rule:*

$$\frac{F, \Box_l^* F, \Gamma \to \Delta}{\Box_l^* F, \Gamma \to \Delta} (\Box_l^* \to)$$

Lemma 3.2.4. *If sequent is derivable in GT_n^c, then it is derivable in $G_1T_n^c$*[1]*.*

Proof. Let $\mathcal{D}$ be a derivation tree of some sequent in GT_n^c. Let's change all the applications of $(\to \Box_l)$ and $(\to \Box^c)$ to applications of $(\to \Box_l)^*$ and $(\to \Box^c)^*$ respectively. Now by changing all the applications of $(\Box_l \to)$ to $(\Box_l \to)^*$ or $(\Box_l^* \to)$ a derivation tree in $G_1T_n^c$ is obtained. The choice of the rule completely depends on the form of the main formula: if the outermost modality is already stared or not. $\Box$

Now to prove that it is possible to transform the derivation tree in the intermediate calculus to derivation tree in the terminating calculus, the admissibility of contraction structural rules must be shown.

Lemma 3.2.5. *The structural rules of contraction are admissible in $G_1T_n^c$.*

Proof. Proof is analogous to the proof of Lemma 1.4.13. $\Box$

Lemma 3.2.6. *If sequent S is derivable in $G_1T_n^c$ and there are no stars in S, then S is derivable in $G^*T_n^c$.*

Proof. Let $\mathcal{D}$ be a derivation tree of sequent S in $G_1T_n^c$. A proof is by induction on the number of applications of $(\Box_l^* \to)$ rule in $\mathcal{D}$. If there are no such applications, then $\mathcal{D}$ is already a proof in $G^*T_n^c$. Otherwise, let's take the top-most application of the rule. Let S_1 be a conclusion and S_2 be a premise of this inference. The main formula of the inference is of the form $\Box_l^* F$ and there are no stars in the initial sequent S, therefore below S_1 there must be an application of $(\Box_l \to)^*$ with the main formula $\Box_l F$. Then $\mathcal{D}$ is of the form:

$$\begin{array}{c}
\mathcal{D}_1 \\
\dfrac{S_2 = F, \Box_l^* F, \Gamma \to \Delta}{S_1 = \Box_l^* F, \Gamma \to \Delta} (\Box_l^* \to) \\
\cdots \\
\dfrac{S_2' = F, \Box_l^* F, \Gamma' \to \Delta'}{S_1' = \Box_l F, \Gamma' \to \Delta'} (\Box_l \to)^* \\
\mathcal{D}_2
\end{array}$$

[1]This lemma proves only that $G_1T_n^c$ is complete. Soundness of this calculus can be proved too, however it is not needed for the completeness proof of the loop-free calculus $G^*T_n^c$.

Moreover, it is possible to find such application of $(\Box_l \to)^*$ to $\Box_l F$, that there would be no applications of $(\to \Box_l)^*$ or $(\to \Box^c)^*$ rule between S_1 and $S_2^{'}$, because any application of those rules removes the outermost stared modality from the antecedent of the sequent. Let $\mathcal{D}_3$ be a path of the derivation from S_1 to $S_2^{'}$. Due to admissibility of contraction, it is possible to apply the rule to $S_2^{'}$ with the main formula F. Let $S_3^{'}$ be a premise of such application. Now by applying all the rules of $\mathcal{D}_3$ to $S_3^{'}$ in the same order sequent S_2 is obtained and derivation tree $\mathcal{D}$ can be replaced by:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ S_2 = F, \Box_l^* F, \Gamma \to \Delta\\ \cdots\\ S_3^{'} = F, F, \Box_l^* F, \Gamma^{'} \to \Delta^{'}\end{array}}{S_2^{'} = F, \Box_l^* F, \Gamma^{'} \to \Delta^{'}}\,(c \to)}{\begin{array}{c}S_1^{'} = \Box_l F, \Gamma^{'} \to \Delta^{'}\\ \mathcal{D}_2\end{array}}\,(\Box_l \to)^*$$

This derivation certainly has one application of $(\Box_l^* \to)$ rule less than derivation tree $\mathcal{D}$. Therefore inductively applying this change it is possible to eliminate all the applications of this rule. $\Box$

From this the completeness of $G^*T_n^c$ follows immediately:

Theorem 3.2.7. *Calculus $G^*T_n^c$ is complete. That is, if some sequent is derivable in GT_n^c, then it is derivable in $G^*T_n^c$.*

Proof. A direct corollary of Lemmas 3.2.4 and 3.2.6. $\Box$

To show that $G^*T_n^c$ is terminating, two more measures are needed. First of all, let's define the one which limits the application of reflexivity rule.

Definition 3.2.8. *Let S be some sequent of some derivation search tree in $G^*T_n^c$. The negative occurrence of $\Box_l$, that is in the scope of negative occurrence of $\Box_{l'}^*$, where $l^{'} = l$ or $l^{'} \neq l$, is called hidden. Otherwise, it is called open. The open modality of sequent S (denoted* $\mathrm{om}(S)$*) is the number of open occurrences of $\Box_l$.*

To define the other measure, indexation of modalities is used. The indexes are part of the terminating calculi for logics $K4_n^c$ and $S4_n^c$, which are presented later (see Definitions 3.5.6 and 3.4.5), however they are not included in $G^*T_n^c$. This time indexes are only needed to prove the finiteness of derivation search in the calculus and they are used in a similar way as later in the proof of Lemma 3.4.12.

Definition 3.2.9. *Let $\mathcal{D}$ be a derivation tree of sequent S in $G^{*}T_n^c$. Let's index all the occurrences of $\Box_l$ of S (for every agent l) with different natural numbers. In order to spread the indexation through the derivation tree $\mathcal{D}$, let's go through all the inferences from bottom to top and let's index all the modalities in all the formulas of every premise of each inference in the same way as the modalities of the respective formulas are indexed in the conclusion. If the occurrence of $\Box_l$ in formula $\Box_l F$ is indexed by i, then it is said that formula $\Box_l F$ is indexed by i.*

For some sequent S_1 of $\mathcal{D}$ let $\Box_l^-(S_1)$ be the set of all the differently indexed formulas or subformulas $\Box_l F$ or $\Box_l^* F$, that occur in S_1 negatively. Let $\Box_l^+(S_1)$ be the set of all the differently indexed formulas or subformulas $\Box_l F$, that occur in S_1 positively. Let $\Box^-(S_1) = \cup_l \Box_l^-(S_1)$ and $\Box^+(S_1) = \cup_l \Box_l^+(S_1)$. A T_n^c-power set of the sequent S_1 (denoted $\mathrm{TPow}(S_1)$) is the set, consisting of all the possible sets of the form $\{F\} \cup \Gamma$, where either (1) $F \in \Box_a^+(S_1)$ and $\Gamma \subseteq \Box_a^-(S_1)$ for some $a \neq c$, or (2) $F \in \Box_c^+(S_1)$ and $\Gamma \subseteq \Box^-(S_1)$.

T_n^c-power of the sequent S_1 (denoted $\mathrm{tp}(S_1)$) is the number of elements in $\mathrm{TPow}(S_1)$.

Notice, that because of the rules of calculus $G^{*}T_n^c$, the occurrences of formula do not change their positiveness (negativeness).

Lemma 3.2.10. *If S_1 and S_2 are part of derivation search tree in calculus GT_n^c and for some inference S_1 is a conclusion and S_2 is some premise, then:*

1. *if the inference is an application of some logical rule then $\mathrm{l}(S_2) < \mathrm{l}(S_1)$, $\mathrm{om}(S_2) \leqslant \mathrm{om}(S_1)$ and $\mathrm{tp}(S_2) \leqslant \mathrm{tp}(S_1)$,*
2. *if the inference is an application of rule $(\Box_l \to)^*$, then $\mathrm{om}(S_2) < \mathrm{om}(S_1)$ and $\mathrm{tp}(S_2) \leqslant \mathrm{tp}(S_1)$,*
3. *if the inference is an application of rule $(\to \Box_l)^*$ or $(\to \Box^c)^*$, then $\mathrm{tp}(S_2) < \mathrm{tp}(S_1)$.*

Proof. In the case of logical rules the proof that $\mathrm{l}(S_2) < \mathrm{l}(S_1)$ is the same as the proof of Lemma 3.1.1. Now, it must be noticed that there is no such rule of calculus $G^{*}T_n^c$ that creates new or differently indexed formulas in the premise compared to the conclusion. Therefore, $\Box_l^+(S_2) \subseteq \Box_l^+(S_1)$ and $\Box_l^+(S_2) \subseteq \Box_l^+(S_1)$ for every l. From this $\mathrm{tp}(S_2) \leqslant \mathrm{tp}(S_1)$ follows immediately. Finally, all the open occurrences of modality in S_1 are either completely removed from S_2 or are also open in S_2. The same can be said about the hidden occurrences of modality. Thus $\mathrm{om}(S_2) \leqslant \mathrm{om}(S_1)$.

In the case of $(\Box_l \to)^*$ rule, the only difference between S_1 and S_2 is that S_1 contains the main formula $\Box_l F$ and S_2 has formulas F and $\Box_l^* F$ instead. All the open occurrences of modality in $\Box_l F$ are part of formula F in S_2, except the outermost occurrence $\Box_l$. Moreover, all the open occurrences of modality in $\Box_l F$ are hidden in $\Box_l^* F$ of S_2, except the outermost occurrence, which is starred instead. From this it can be concluded, that $\mathrm{om}(S_2) < \mathrm{om}(S_1)$. The proof that $\mathrm{tp}(S_2) \leqslant \mathrm{tp}(S_1)$ is analogous to the case of logical rules.

In the case of rule $(\to \Box_l)^*$ or $(\to \Box^c)^*$, it is also easy to see that $\Box_l^+(S_2) \subseteq \Box_l^+(S_1)$ and $\Box_l^+(S_2) \subseteq \Box_l^+(S_1)$ for every l. Moreover, if the main formula of the application $\Box_l F$ is not part of S_2, then the set $\{\Box_l F\}$ is part of $\mathrm{TPow}(S_1)$, but $\{\Box_l F\} \notin \mathrm{TPow}(S_2)$ and $\mathrm{tp}(S_2) < \mathrm{tp}(S_1)$. Otherwise, if $\Box_l F$ is part of S_2 and $l \neq c$, then it must also be a subformula of some formula $G' = \Box_l G''$ in an antecedent of S_1. In the case of $l = c$, the antecedent of S_1 must contain formula $G' = \Box_{l'} G''$. In any case let G bet the longest such formula in the antecedent of S_1. The set $\{\Box_l F, G\}$ is part of $\mathrm{TPow}(S_1)$, however it is not part of $\mathrm{TPow}(S_2)$, because formula G is definitely not part of S_2. Thus, once again, $\mathrm{tp}(S_2) < \mathrm{tp}(S_1)$. $\Box$

Lemma 3.2.11. *In any derivation search tree of $G^*T_n^c$ if for some sequent S:*

1. $\mathrm{l}(S) = 0$*, then it is not possible to apply any logical rule to* S.
2. $\mathrm{om}(S) = 0$*, then it is not possible to apply rule* $(\Box_l \to)^*$ *to* S.
3. $\mathrm{tp}(S) = 0$*, then it is not possible to apply rule* $(\to \Box_l)^*$ *or* $(\to \Box^c)^*$ *to* S.

Proof. The proof of the first part is analogous to the proof of Lemma 3.1.2.

If $\mathrm{om}(S) = 0$, then there are no open modalities in sequent S. Therefore, there is no formula of the form $\Box_l F$ in the antecedent of S and the rule $(\Box_l \to)^*$ cannot be applied to S.

Finally, if $\mathrm{tp}(S) = 0$, then $\Box^+(S) = \varnothing$ and therefore, the succedent of S does not contain formula of the form $\Box_l F$. Consequently, neither rule $(\to \Box_l)^*$ nor rule $(\to \Box^c)^*$ can be applied to S. $\Box$

Once again, direct corollary of Lemmas 3.2.10 and 3.2.11 is that $G^*T_n^c$ is terminating.

Theorem 3.2.12. *Calculus $G^*T_n^c$ is terminating.*

Proof. The reasoning is the same as in the proof of Theorem 3.1.3. □

3.3 Logic *S4*

A loop problem in calculus *GS4* is analysed in detail in [4]. In that article, a solution — terminating calculus for *S4* — is presented. In this dissertation this calculus is presented only to make the references to it more clear.

In the calculus different labels are used to obtain derivation search termination. First of all, after application of reflexivity rule the outermost modality is starred in the same way, as in the case of calculus $G^*T_n^c$. In addition, to avoid excess applications of the reflexivity rule, not only the main formula of the application is starred, but also any occurrence of the main formula as subformula of other formulas of the sequent. For this purpose all the negative occurrences of $\Box$ are indexed.

In order to note at what place in the derivation search tree the formula was introduced to the sequent, formula numeration is used. The formula is put in square brackets and its number is written as an index in the top right, for example, $[\Box p \wedge q]^1$. In fact only formulas, that were introduced after some application of transitivity rule, are important, so new number is introduced only after the application of transitivity rule.

Positive occurrences of $\Box$ are indexed too, because all the possible main formulas of transitivity rule must be identified in order to check if the rule was applied to them earlier. However as offered in [45] the indexation is limited to positive occurrences of $\Box$ that are in the scope of negative occurrence of $\Box$, because otherwise the main formula of application of transitivity rule can not reappear in the sequent. Integer starting from 1 is used for indexes of both (positive and negative) occurrences of $\Box$. Indexes are written in the top right corner of the modality symbol but to avoid confusion the negative indexes are preceded with $\ominus$, for example $\Box^1, \Box^{\ominus 1}$.

Finally to note if the transitivity rule was applied to the formula earlier and at what place in the derivation search tree it was done for the last time, marks are used. They are written in brackets just after the index, for example $\Box^{5(3)}$. If no mark is used brackets are omitted too. Only positive occurrences of $\Box$ can be marked and there is no need to mark not indexed occurrences of $\Box$.

Definition 3.3.1. *A sequent is called labelled*[2] *for S4, if every positive occurrence of* $\Box$ *that is in the scope of negative occurrence of* $\Box$ *is indexed with integer, every negative occurrence of* $\Box$ *is either indexed with indexes of the form* $\ominus i$ *or starred and all the formulas are numbered. Multiset of formulas* Γ *is labelled for S4, if* $\Gamma \to$ *or* $\to \Gamma$ *is a labelled for S4 sequent. Formula* F *is labelled for S4, if* $\{F\}$ *is labelled for S4 multiset.*

labelled expressions are denoted in the same way as non-labelled ones: letter S is used for sequents, capital Latin letters are used for formulas and capital Greek letters for multisets of formulas. It is usually clear from the context if the notation means labelled or non-labelled expression.

Some initial labels need to be given to the initial sequent.

Definition 3.3.2. *Labelling for S4 of sequent* S *is denoted* $\mathrm{Lab}_{S4}(S)$ *and labelled for S4 sequent* $\mathrm{Lab}_{S4}(S)$ *is obtained from* S *by (1) indexing all the positive occurrences of* $\Box$ *that are in the scope of negative occurrence of* $\Box$ *with different natural numbers, (2) indexing all the negative occurrences of* $\Box$ *with different indexes of the form* $\ominus i$ *and (3) attaching number* 1 *to every formula. No marks are needed in* $\mathrm{Lab}_{S4}(S)$.

A sequent calculus that employs labelling, is called *labelled sequent calculus*. It differs from the regular one in two aspects. Firstly, all the derivation search trees in labelled sequent calculus consist of labelled sequents only. Secondly, if S is the initial sequent of the derivation search tree in labelled sequent calculus, then it must be obtained from some regular sequent by labelling.

To compare two formulas without taking into account their labels one more definition is used.

Definition 3.3.3. *A projection of labelled formula* F *(denoted* $\mathrm{Proj}(F)$*) is obtained by removing all the indexes, marks, numbers and stars from the formula* F.

Now the definition of a loop-free sequent calculus for *S4* can be provided.

[2]In [4] the term “indexed sequent” (“indexation”, “indexed calculus”) is used. However, in the review of the article, term “label” was offered for all the stars, positive and negative indexes, marks and formula numbers. Although this term was not included in the article, it is used here. It is better to say “labelled sequent” (“labelling”, “labelled calculus”), because such sequent contains indexes as well as formula numbers and possibly stars and marks.

Definition 3.3.4. *The labelled sequent calculus without loops for logic S4 (G*S4) consists of axiom* $\Gamma, [F_1]^n \to [F_2]^n, \Delta$*, where* $\mathrm{Proj}(F_1) = \mathrm{Proj}(F_2)$ *and rules:*

Negation:

$$\frac{\Gamma \to \Delta, [F]^n}{[\neg F]^n, \Gamma \to \Delta}\ (\neg \to) \qquad \frac{[F]^n, \Gamma \to \Delta}{\Gamma \to \Delta, [\neg F]^n}\ (\to \neg)$$

Conjunction:

$$\frac{[F]^n, [G]^n, \Gamma \to \Delta}{[F \wedge G]^n, \Gamma \to \Delta}\ (\wedge \to) \qquad \frac{\Gamma \to \Delta, [F]^n \qquad \Gamma \to \Delta, [G]^n}{\Gamma \to \Delta, [F \wedge G]^n}\ (\to \wedge)$$

Disjunction:

$$\frac{[F]^n, \Gamma \to \Delta \qquad [G]^n, \Gamma \to \Delta}{[F \vee G]^n, \Gamma \to \Delta}\ (\vee \to) \qquad \frac{\Gamma \to \Delta, [F]^n, [G]^n}{\Gamma \to \Delta, [F \vee G]^n}\ (\to \vee)$$

Implication:

$$\frac{\Gamma \to \Delta, [F]^n \qquad [G]^n, \Gamma \to \Delta}{[F \supset G]^n, \Gamma \to \Delta}\ (\supset\to) \qquad \frac{[F]^n, \Gamma \to \Delta, [G]^n}{\Gamma \to \Delta, [F \supset G]^n}\ (\to\supset)$$

Reflexivity:

$$\frac{[F]^n, [\Box^* F]^n, \Gamma^{\ominus i *} \to \Delta^{\ominus i *}}{[\Box^{\ominus i} F]^n, \Gamma \to \Delta}\ (\Box^{\ominus i} \to)$$

where $\Gamma^{\ominus i *}$ *(*$\Delta^{\ominus i *}$*) is obtained from* Γ *(respectively* Δ*) by replacing all the occurrences of* $\Box^{\ominus i}$ *with* $\Box^*$*.*
Transitivity:

$$\frac{[\Gamma^{i \leftarrow n}]^n, \Box^* \Gamma^{i \leftarrow n} \to [F]^n}{\Box^* \Gamma, \Sigma_1 \to \Sigma_2, \Box \Delta, [\Box^{i(m)} F]^{n-1}}\ (\to \Box^i)$$

where Σ_1 *and* Σ_2 *are empty or consist of propositional variables only,* i *is some index or nothing (denoted* $i = \varnothing$*),* (m) *is some mark or nothing (denoted* $(m) = \varnothing$*). If* $\Gamma = [G_1]^{n_1}, \ldots, [G_k]^{n_k}$ *then* $[\Gamma]^n = [G_1]^n, \ldots, [G_k]^n$*. If* $i = \varnothing$*, then* $\Gamma^{i \leftarrow n} = \Gamma$*, and if* $i \neq \varnothing$*, then* $\Gamma^{i \leftarrow n}$ *is obtained from* Γ *by replacing all the occurrences of* $\Box^{i(m)}$ *with* $\Box^{i(n)}$*. What is more, transitivity rule can only be applied if either* $(m) = \varnothing$ *or* $\Box^* \Gamma$ *contains at least one formula of the form* $[\Box^* H]^l$ *where* $m < l \leqslant n - 1$*.*

Simplification:

$$\frac{[\Box^* F]^{n_1}, \Gamma \to \Delta}{[\Box^* F]^{n_1}, [\Box^* F]^{n_2}, \Gamma \to \Delta} \, (\Box^{n_1, n_2} \to)$$

where $n_1 < n_2$.
If possible, simplification rule must be applied first.

The soundness, completeness of *G*S4* and the fact that the calculus is terminating are proved in [4]. Moreover in the article the development of the calculus from the one provided in [37] is described.

3.4 Logic $S4_n^c$

The source of loops in calculi $GS4_n$ and $GS4_n^c$ is essentially the same as in *GS4*, therefore similar techniques to the ones used to develop calculus *G*S4* can be applied. However, only logic $S4_n^c$ is analysed, because such is the aim of this dissertation. This section is divided into several parts. In Subsection 3.4.1 the Gentzen-type calculus for $S4_n^c$ is presented. In Subsection 3.4.2 finiteness of derivation search in the calculus is proved and in Subsection 3.4.3 soundness and completeness is demonstrated.

3.4.1 The Calculus

In calculus $GS4_n^c$ there are two similar rules: $(\to \Box_l)$ (transitivity rule) and $(\to \Box^c)$ (central agent rule). They both cause similar problems with cycles, therefore to get the terminating calculus, they both are altered in similar way. Thus to make the discussion clearer and shorter, both rules are called *succedental*.

In spite of all the similarities, calculus $GS4_n^c$ differs from *GS4* in one minor although important aspect. In the derivation search trees of the monomodal calculus formulas, that start with modality, cannot disappear from the antecedent of the sequent. That is if such formula is part of the antecedent of the conclusion of some inference, then it is definitely part of the antecedent of every premise of the inference. This is not the case in the multimodal calculus, because the application of rule $(\to \Box_l)$ wipes out from the antecedent all the formulas that start with modality $\Box_{l'}$, where $l' \neq l$.

Nevertheless the same labels as in *G*S4* are used and their meaning is very similar, however usage usually differs in some aspects. First of all, sometimes in $GS4_n^c$ it is necessary to apply reflexivity rule to the same formula more than once. The situation is explained in the following example.

Example 3.4.1. *Consider the derivation tree of* $\Box_1\Box_2 p \to \Box_1 p$ *in* $GS4_n^c$*:*

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{S_6 = p, \Box_2 p, \Box_1\Box_2 p \to p}{S_5 = \Box_2 p, \Box_1\Box_2 p \to p}\,(\Box_2 \to)}{S_4 = \Box_1\Box_2 p \to p}\,(\Box_1 \to)}{S_3 = p, \Box_2 p, \Box_1\Box_2 p \to \Box_1 p}\,(\to \Box_1)}{S_2 = \Box_2 p, \Box_1\Box_2 p \to \Box_1 p}\,(\Box_2 \to)}{S_1 = \Box_1\Box_2 p \to \Box_1 p}\,(\Box_1 \to)
$$

The derivation is meant to be as similar to derivations of calculus G^*S4 as possible, therefore transitivity rule is applied only when the application of reflexivity rule produces a loop. In the derivation rules $(\Box_1 \to)$ *and* $(\Box_2 \to)$ *are applied twice to the same formula, however the second application of rule* $(\Box_1 \to)$ *can be avoided by replacing the application of* $(\to \Box_1)$ *rule to G^*S4-style transitivity rule. On the other hand, even after the change* $(\Box_2 \to)$ *rule must be applied for the second time, because after the application of transitivity rule, formula* $\Box_2 p$ *disappears from the antecedent of the sequent.*

To deal with this situation, after application of reflexivity rule, to formula $\Box_l^{\ominus i} F$ all the occurrences of negative index $\ominus i$ are starred, however contrary to the G^*S4 case the index itself is not removed. To shorten the notation starred negative indexes are denoted by changing the symbol $\ominus$ to the star: $\Box_l^{*i} F$. However these stars are needed only until some inference removes the formula $\Box_l^{*i} F$ from the antecedent. Therefore, after application of l'-transitivity rule, all the stared negative modalities of all the agents, other than l', are changed back to regular negative modalities.

Next, the usage of marks must also be altered. The logic behind the marks in G^*S4 is based on the fact, that formulas that start with modality, cannot disappear from the antecedent of the sequent. In fact, this is true only to formulas of the form $\Box_l F$, when l-transitivity rule is applied, and to all the formulas that start with modality, when central agent rule is applied. Therefore, marks for modality $\Box_l$ are meaningful, only as long as no other transitivity rule, except $(\to \Box_l)$ is applied. Thus, after application of l-transitivity rule, the marks of modalities of any agent except l should be removed. However central agent rule should leave them as they are.

Now the positive indexes of the modalities must be revised. All the formulas, that can be the main formulas of the application of succedental rule more than once, must have their outermost modality indexed. Moreover, the proof that G^*S4 is terminating in [4] uses the fact, that non-indexed modality disappears from the derivation search tree after application of succedental rule to it. Although this is not necessary, it eases the proof that $G^*S4_n^c$ is terminating too. Having this in mind, it is defined, which modalities should be indexed.

Definition 3.4.2. *Let S be some sequent and $\Box_l F$ a subformula of some formula of S. If $\Box_l F$ occurs in S positively, then occurrence of $\Box_l$ is called indexable, if:*

- *$l \neq c$ and $\Box_l$ is in the scope of negative occurrence of $\Box_l$,*
- *$l = c$ and $\Box_l$ is in the scope of negative occurrence of $\Box_{l'}$ for some l'.*

This definition takes into account the properties discussed earlier. However, by dropping the unnecessary requirements, the stricter definition of indexable occurrence of $\Box_l$ may be developed.

Now the definition of indexable occurrence of $\Box_l$ is used to define labelled sequent and labelling for $S4_n^c$.

Definition 3.4.3. *A sequent is called labelled for $S4_n^c$, if every indexable occurrence of $\Box_{l_1}$ is indexed with integer, every negative occurrence of $\Box_{l_2}$ is either indexed with indexes of the form $\ominus i$ or $*i$ and all the formulas are numbered. Multiset of formulas Γ is labelled for $S4_n^c$, if $\Gamma \rightarrow$ or $\rightarrow \Gamma$ is a labelled for $S4_n^c$ sequent. Formula F is labelled for $S4_n^c$, if $\{F\}$ is labelled for $S4_n^c$ multiset.*

Definition 3.4.4. *Labelling for $S4_n^c$ of sequent S is denoted* $\mathrm{Lab}(S)$ *and labelled for $S4_n^c$ sequent* $\mathrm{Lab}(S)$ *is obtained from S by (1) indexing all the indexable occurrences of $\Box_{l_1}$ with different natural numbers, (2) indexing all the negative occurrences of $\Box_{l_2}$ with different indexes of the form $\ominus i$ and (3) attaching number* 1 *to every formula. No marks are needed in* $\mathrm{Lab}(S)$.

Once again it must be noted that derivation search trees in labelled calculi consists of labelled sequents and their initial sequent is obtained by labelling the regular sequent.

Finally, having two succedental rules (compared to only transitivity rule in G^*S4) causes one more problem. Positive occurrences of formulas $\Box_c F$ can be main formulas of application of both succedental rules. In this case it would be important to also note which rule (if any) was applied to this formula earlier. However, this situation is avoided by restricting the application of transitivity rule: the rule $(\to \Box_c)$ cannot be applied (only rule $(\to \Box_a)$, where $a \neq c$). It is not hard to see, that this restriction does not have any effect on soundness and completeness. However now it is obvious that if some rule was applied to the positive occurrence of $\Box_l F$, then if $l = c$, it was central agent rule, and otherwise it was transitivity rule.

Now the definition of a loop-free sequent calculus for $S4_n^c$ can be provided.

Definition 3.4.5. *The labelled sequent calculus without loops for logic $S4_n^c$ ($G^*S4_n^c$) consists of axiom $\Gamma, [F_1]^n \to [F_2]^n, \Delta$, where $\mathrm{Proj}(F_1) = \mathrm{Proj}(F_2)$, the same logical rules as in G^*S4 and rules:*

Reflexivity:

$$\frac{[F]^n, [\Box_l^{*i} F]^n, \Gamma^{\ominus i*} \to \Delta^{\ominus i*}}{[\Box_l^{\ominus i} F]^n, \Gamma \to \Delta} \; (\Box_l^{\ominus i} \to)$$

where $\Gamma^{\ominus i*}$ ($\Delta^{\ominus i*}$) is obtained from Γ (respectively Δ) by replacing all the occurrences of $\Box_l^{\ominus i}$ with $\Box_l^{*i}$.
Transitivity:

$$\frac{\left\{[\Gamma_1^{i\leftarrow n}]^n, \Box_a^* \Gamma_1^{i\leftarrow n} \to [F]^n\right\}^{\neq a:\not{*},\not{()}}}{\Box_a^* \Gamma_1, \Box_{\neq a}^* \Gamma_2, \Sigma_1 \to \Sigma_2, \Box_* \Delta, [\Box_a^{i(m)} F]^{n-1}} \; (\to \Box_a^i)$$

where Σ_1 and Σ_2 are empty or consist of propositional variables only, $a \neq c$, i is some index or nothing (denoted $i = \varnothing$), (m) is some mark or nothing (denoted $(m) = \varnothing$). If $\Gamma_1 = [G_1]^{n_1}, \ldots, [G_k]^{n_k}$ then $[\Gamma_1]^n = [G_1]^n, \ldots, [G_k]^n$. If $i = \varnothing$, then $\Gamma_1^{i\leftarrow n} = \Gamma_1$, and if $i \neq \varnothing$, then $\Gamma_1^{i\leftarrow n}$ is obtained from Γ_1 by replacing all the occurrences of $\Box_l^{i(m)}$ with $\Box_l^{i(n)}$. In addition, $\Box_a^* \Gamma_2$ consists of formulas of the form $\Box_a^{*j} G$, $\Box_{\neq a}^* \Gamma_2$ consists of formulas of the form $\Box_l^{*j} G$, where $l \neq a$ and sequent $\{S\}^{\neq a:\not{*},\not{()}}$ is obtained from S by replacing all the occurrences of $\Box_l^{*j}$ to $\Box_l^{\ominus j}$ and $\Box_l^{j(m')}$ to $\Box_l^j$ for every $l \neq a$, every j and every m'.
What is more, transitivity rule can only be applied if either $(m) = \varnothing$ or $\Box_a^* \Gamma_1$ contains at least one formula of the form $[\Box_a^{*j} H]^{n_1}$ where $m < n_1 \leqslant n - 1$. Formula of the form $[\Box_a^{*j} H]^{n_1}$ is called necessity formula with number n_1.
Central agent:

$$\frac{[\Gamma^{i\leftarrow n}]^n, \Box_*^* \Gamma^{i\leftarrow n} \to [F]^n}{\Box_*^* \Gamma, \Sigma_1 \to \Sigma_2, \Box_* \Delta, [\Box_c^{i(m)} F]^{n-1}} \; (\to \Box^{c,i})$$

where Σ_1 and Σ_2 are empty or consist of propositional variables only, i is some index or $i = \varnothing$, (m) is some mark or $(m) = \varnothing$.
Once again, central agent rule can only be applied if either $(m) = \varnothing$ or $\Box_*^*\Gamma$ contains at least one formula of the form $[\Box_l^{*j}H]^{n_1}$ for some l, where $m < n_1 \leqslant n-1$.
Simplification:

$$\frac{[\Box_l^{*i}F]^{n_1}, \Gamma \to \Delta}{[\Box_l^{*i}F]^{n_1}, [\Box_l^{*i}F]^{n_2}, \Gamma \to \Delta}\ (\Box_l^{n_1,n_2} \to)$$

where $n_1 < n_2$.
If possible, simplification rule must be applied first.

First let's show that indexable modality has the desired properties.

Lemma 3.4.6. *Let $\mathcal{D}$ be a derivation search tree in $G^*S4_n^c$. Let S_1 be a conclusion of application of succedental rule to formula $\Box_l F$ in $\mathcal{D}$. Moreover, let S_2 be a premise of the inference. (1) If this occurrence of $\Box_l$ is not indexable in S_1, then S_2 does not contain formula, which is a superformula of $\Box_l F$. (2) Let $\mathcal{B}$ be a branch, which contains S_1 and S_2. If there is another application of succedental rule to $\Box_l F_1$, where $\mathrm{Proj}(F_1) = \mathrm{Proj}(F)$, in $\mathcal{B}$ above S_2, then this occurrence of $\Box_l$ is indexable in S_1.*

Proof. Let's leave labels out, because they are not important for this proof. Let $l \neq c$ and $\mathrm{Proj}(S_1) = \Box_l\Gamma_1, \Box_{\neq l}\Gamma_2, \Sigma_1 \to \Sigma_2, \Box_*\Delta, \Box_l F$. In this case $\mathrm{Proj}(S_2) = \Gamma_1, \Box_l\Gamma_1 \to F$. Let G be a superformula of $\Box_l F$ in S_2. Now either $G = \Box_l G'$ and it is part of antecedent of S_1, or $\Box_l G$ is part of antecedent of S_1. In both ways $\Box_l F$ is in the scope of negative occurrence of $\Box_l$ in S_1 and therefore this occurrence of $\Box_l$ is indexable. This is a contradiction, therefore there are no superformulas of $\Box_l F$ in S_2. The case of $l = c$ is analogous.

To prove the second part, the first part can be used. If this occurrence of $\Box_l$ is not indexable, then there are no superformula of $\Box_l F$ in S_2 and therefore there are no superformula of $\Box_l F$ in $\mathcal{B}$ above S_2 (once again, labels are ignored here). Thus it is impossible to apply succedental rule to $\Box_l F_1$, where $\mathrm{Proj}(F_1) = \mathrm{Proj}(F)$, above S_2. Once again a contradiction is obtained and it must be concluded that $\Box_l$ is indexable. $\Box$

Next some properties of $G^*S4_n^c$ must be demonstrated.

Lemma 3.4.7. *For every labelled sequent S of any derivation search tree in $G^{*}S4_n^c$ the following are correct:*

1. *In application of any rule except simplification to the sequent S the number of the main formula of the application is always the largest among the numbers of S.*

2. *In application of simplification rule to the sequent S the number of one main formula (the one which is omitted in the premise) is largest among the numbers of S.*

3. *The application of succedental rule in S introduces a number that is larger than any number in S.*

4. *If n is the largest number of S, then all the formulas are numbered n, except the ones of the form $\Box_l^{*i}F$, that* ***can*** *be numbered lower.*

Proof. All of this follows from noticing that (1) all the formulas start with number 1 (2) only succedental rules can introduce new number (3) after application of these rules only formulas of the form $\Box_l^{*i}F$ keep their numbers, all the other formulas get the new one and (4) no rule except simplification can be applied to the formula of the form $\Box_l^{*i}F$. By induction it can be shown that the new number is always the largest number in the premise. $\Box$

Now several examples are presented.

Example 3.4.8. *First of all, let's show, that sequent $\Box_1\Box_2 p \to \Box_1 p$ is derivable in $G^{*}S4_n^c$. The labelling results in $[\Box_1^{\ominus 1}\Box_2^{\ominus 2}p]^1 \to [\Box_1 p]^1$:*

$$
\dfrac{\dfrac{\dfrac{\dfrac{S_5 = [p]^2, [\Box_2^{*2}p]^2, [\Box_1^{*1}\Box_2^{*2}p]^1 \to [p]^2}{S_4 = [\Box_2^{\ominus 2}p]^2, [\Box_1^{*1}\Box_2^{\ominus 2}p]^1 \to [p]^2}\,(\Box_2^{\ominus 2}\to)}{S_3 = [p]^1, [\Box_2^{*2}p]^1, [\Box_1^{*1}\Box_2^{*2}p]^1 \to [\Box_1 p]^1}\,(\to\Box_1)}{S_2 = [\Box_2^{\ominus 2}p]^1, [\Box_1^{*1}\Box_2^{\ominus 2}p]^1 \to [\Box_1 p]^1}\,(\Box_2^{\ominus 2}\to)}{S_1 = [\Box_1^{\ominus 1}\Box_2^{\ominus 2}p]^1 \to [\Box_1 p]^1}\,(\Box_1^{\ominus 1}\to)
$$

Example 3.4.9. *This example demonstrates, that calculus $G^{*}S4_n^c$ allows application of succedental rule to the same formula more than once, if needed.*

The original sequent is $\Box_1 \neg \Box_c (p \vee \Box_1 \neg \Box_2 p) \to$ and labelling results in $[\Box_1^{\ominus 1} \neg \Box_c^1 (p \vee \Box_1^2 \neg \Box_2^{\ominus 2} p)]^1 \to$. The derivation is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{S_6 = [p]^4, [\Box_2^{*2} p]^3, [\Box_1^{*1} \neg \Box_c^{1(4)} (p \vee \Box_1^{2(3)} \neg \Box_2^{*2} p)]^1 \to [p]^4, [\Box_1^{2(3)} \neg \Box_2^{*2} p]^4, [\Box_c^{1(4)} (p \vee \Box_1^{2(3)} \neg \Box_2^{*2} p)]^4}{S_5 = [p]^3, [\Box_2^{*2} p]^3, [\Box_1^{*1} \neg \Box_c^{1} (p \vee \Box_1^{2(3)} \neg \Box_2^{*2} p)]^1 \to [\Box_c^{1} (p \vee \Box_1^{2(3)} \neg \Box_2^{*2} p)]^3}\,(\to \Box^{c,1})}{S_4 = [\Box_2^{\ominus 2} p]^3, [\Box_1^{*1} \neg \Box_c^{1} (p \vee \Box_1^{2(3)} \neg \Box_2^{\ominus 2} p)]^1 \to [\Box_c^{1} (p \vee \Box_1^{2(3)} \neg \Box_2^{\ominus 2} p)]^3}\,(\Box_2^{\ominus 2} \to)}{S_3 = [\Box_1^{*1} \neg \Box_c^{1(2)} (p \vee \Box_1^{2} \neg \Box_2^{\ominus 2} p)]^1 \to [p]^2, [\Box_1^{2} \neg \Box_2^{\ominus 2} p)]^2, [\Box_c^{1(2)} (p \vee \Box_1^{2} \neg \Box_2^{\ominus 2} p)]^2}\,(\to \Box_1^2)}{S_2 = [\Box_1^{*1} \neg \Box_c^{1} (p \vee \Box_1^{2} \neg \Box_2^{\ominus 2} p)]^1 \to [\Box_c^{1} (p \vee \Box_1^{2} \neg \Box_2^{\ominus 2} p)]^1}\,(\to \Box^{c,1})}{S_1 = [\Box_1^{\ominus 1} \neg \Box_c^{1} (p \vee \Box_1^{2} \neg \Box_2^{\ominus 2} p)]^1 \to}\,(\Box_1^{\ominus 1} \to), (\neg \to)$$

To shorten the derivation some applications of logical rules are left out.

It can be noticed that central agent rule can be applied to the sequent S_2, because the occurrence of $\Box_c^1$ is not marked. However the same rule with the same main formula can not be applied to S_3, because $\Box_c^1$ is now marked and there is no such number that can satisfy the condition $2 < n \leqslant 2$. So another main formula must be chosen, and the only alternative is the one with unmarked occurrence of $\Box_1^2$. What is more, the rule $(\to \Box^{c,1})$ can be applied for the second time to the sequent S_5. This is because once again $\Box_c^1$ is not marked. It is clear that S_6 is an axiom and thus this is a derivation tree of sequent S_1.

3.4.2 Finiteness of Derivation Search in the Calculus

First of all let's show that every derivation search tree in calculus $G^{*}S4_n^c$ is finite. Several measures are used, and the first one is the length of the sequent. It obviously decreases, when logical or simplification rule is applied.

Next the *starless modality*[3] of labelled sequent S is defined to be the number of different not starred negative indexes in the sequent and denoted $\mathrm{sm}(S)$. If derivation provided in Example 3.4.9 is considered, then $\mathrm{sm}(S_1) = 2$, $\mathrm{sm}(S_2) = \mathrm{sm}(S_3) = \mathrm{sm}(S_4) = 1$, $\mathrm{sm}(S_5) = \mathrm{sm}(S_6) = 0$. The starless modality decreases, when reflexivity rule is applied to the sequent.

Finally the power of a sequent is adapted to the multimodal logic and another measure is defined.

Definition 3.4.10. *For some labelled sequent S the set of all formulas and subformulas of the form $\Box_l^{\ominus i} F$ or $\Box_l^{*i} F$ without their number that are part of S is denoted $\Box_l^{-}(S)$. The set of all formulas and subformulas of the form $\Box_l F$, $\Box_l^i F$ or $\Box_l^{i(m)} F$ without their number that are part of S is denoted*

[3] In [4] the term "negative modality" was used. However in the multimodal case another measure is needed, for which this term suits better. Therefore, this measure is renamed.

$\Box_l^+(S)$. $\Box^-(S) = \cup_l \Box_l^-(S)$ *and* $\Box^+(S) = \cup_l \Box_l^+(S)$. *The negative modality of sequent* S *(denoted* $\mathrm{nm}(S)$*) is the number of elements in* $\Box^-(S)$.

The set $\mathrm{P_{ref}}(l, n, S)$ *consists of all the formulas* F*, such that* $F = \Box_l^{\ominus i} G$ *or* $F = \Box_l^{*i} G$ *and* $[F]^k, k \leqslant n$ *occurs as the formula of the antecedent of* S.[4] $\mathrm{P_{ref}}(n, S) = \cup_l \mathrm{P_{ref}}(l, n, S)$.

A power set of labelled sequent S *(denoted* $\mathrm{Pow}(S)$*) is a set, consisting of all the possible sets of the form* $\{F\} \cup \Gamma$ *that meet the following requirements:*

1. $F \in \Box_a^+(S)$ *for some* a $(a \neq c)$,
2. $\Gamma \subseteq \Box_a^-(S)$,
3. *One of the following holds:*
   (a) *The outmost occurrence of* $\Box_a$ *in* F *is not marked in* S.
   (b) *The outmost occurrence of* $\Box_a$ *in* F *is marked with* m *in* S *and* $\Gamma \not\subseteq \mathrm{P_{ref}}(a, m, S)$.

A central power set of labelled sequent S *(denoted* $\mathrm{CPow}(S)$*) is a set, consisting of all the possible sets of the form* $\{F\} \cup \Gamma$ *that meet the following requirements:*

1. $F \in \Box_c^+(S)$,
2. $\Gamma \subseteq \Box^-(S)$,
3. *One of the following holds:*
   (a) *The outmost occurrence of* $\Box_c$ *in* F *is not marked in* S.
   (b) *The outmost occurrence of* $\Box_c$ *in* F *is marked with* m *in* S *and* $\Gamma \not\subseteq \mathrm{P_{ref}}(m, S)$.

Power of the sequent S *(denoted* $\mathrm{p}(S)$*) is the number of elements in* $\mathrm{Pow}(S)$. *Central power of the sequent* S *(denoted* $\mathrm{c}(S)$*) is the number of elements in* $\mathrm{CPow}(S)$.*elements in* $\mathrm{CPow}(S)$.

Example 3.4.11. *If sequent* S_3 *from Example 3.4.9 is considered, then:*

- $\Box^+(S_3) = \{\Box_c^{1(2)}(p \vee \Box_1^2 \neg \Box_2^{\ominus 2} p), \Box_1^2 \neg \Box_2^{\ominus 2} p\}$, *let's denote the formulas of the set as* F_1 *and* F_2 *respectively. Then* $\Box_c^+(S_3) = \{F_1\}$ *and* $\Box_1^+(S_3) = \{F_2\}$.

[4] The notation $\mathrm{P_{ref}}(l, n, S)$ is due to [37]. In that article $\mathrm{P}(n)$ is used for logic *K4* and by adapting it to this article $\mathrm{P}(l, n, S)$ could be defined in the same way, except that condition $k \leqslant n$ have to be changed to $k < n$.

- $\Box^{-}(S_3) = \{\Box_1^{*1}\neg\Box_c^{1(2)}(p\vee\Box_1^{2}\neg\Box_2^{\ominus 2}p), \Box_2^{\ominus 2}p\}$ *and obviously* $\mathrm{nm}(S_3) = 2$. *Let's denote the formulas of the set as* G_1 *and* G_2 *respectively. Then* $\Box_1^{-}(S_3) = \{G_1\}$ *and* $\Box_2^{-}(S_3) = \{G_2\}$.
- $\mathrm{P_{ref}}(2, S_3) = \{\Box_1^{*1}\neg\Box_c^{1(2)}(p \vee \Box_1^{2}\neg\Box_2^{\ominus 2}p)\} = \{G_1\}$.
- $\mathrm{CPow}(S_3) = \Big\{\{F_1, G_2\}, \{F_1, G_1, G_2\}\Big\}$, *so* $\mathrm{c}(S_3) = 2$.
- $\mathrm{Pow}(S_3) = \Big\{\{F_2\}, \{F_2, G_1\}\Big\}$, *so* $\mathrm{p}(S_3) = 2$.

The central power of the sequent does not increase after application of logical, simplification or reflexivity rules, but it decreases after application of central agent rule. Similarly, the power and negative modality of the sequent does not increase after application of logical, simplification, reflexivity or central agent rules, but at least on of them decreases after application of transitivity rule. This is shown in a proof of the following lemma.

Lemma 3.4.12. *If* S_1 *and* S_2 *are part of derivation search tree in* $G^{*}S4_n^{c}$ *and for some inference* S_1 *is a conclusion and* S_2 *is a premise then:*

1. *If the inference is application of some logical or simplification rule, then* $\mathrm{l}(S_2) < \mathrm{l}(S_1)$, $\mathrm{sm}(S_2) \leqslant \mathrm{sm}(S_1)$, $\mathrm{c}(S_2) \leqslant \mathrm{c}(S_1)$, $\mathrm{p}(S_2) \leqslant \mathrm{p}(S_1)$ *and* $\mathrm{nm}(S_2) \leqslant \mathrm{nm}(S_1)$.
2. *If the inference is application of reflexivity rule, then* $\mathrm{sm}(S_2) < \mathrm{sm}(S_1)$, $\mathrm{c}(S_2) \leqslant \mathrm{c}(S_1)$, $\mathrm{p}(S_2) \leqslant \mathrm{p}(S_1)$ *and* $\mathrm{nm}(S_2) \leqslant \mathrm{nm}(S_1)$.
3. *If the inference is application of central agent rule, then* $\mathrm{c}(S_2) < \mathrm{c}(S_1)$, $\mathrm{p}(S_2) \leqslant \mathrm{p}(S_1)$ *and* $\mathrm{nm}(S_2) \leqslant \mathrm{nm}(S_1)$.
4. *If the inference is application of transitivity rule, then either* $\mathrm{p}(S_2) < \mathrm{p}(S_1)$ *and* $\mathrm{nm}(S_2) \leqslant \mathrm{nm}(S_1)$ *or* $\mathrm{nm}(S_2) < \mathrm{nm}(S_1)$.

Proof. The four parts are proved separately.

1. From the form of logical rules it can be seen that after the application the main formula loses at least one logical operator so the length of the main formula decreases. Moreover, the other formulas remain unchanged so the length of the premise must be smaller than the length of the conclusion.

   In the case of simplification rule it is enough to spot that the premise has one formula less than the conclusion and that the formula, that is omitted in the premise, has at least one logical operator — $\Box_l^{*}$. So obviously in both cases $\mathrm{l}(S_2) < \mathrm{l}(S_1)$.

Next, it is obvious that in the entire derivation search tree new negative index is not produced. What is more, starless modality of the sequent can get smaller after application of some logical rule, because the main formula of the application can split into two premises and some negative indexes could appear only in one of the premises[5]. Finally, in the application of logical or simplification rule stared negative indexes do not lose their stars. So it must be concluded that $\mathrm{sm}(S_2) \leqslant \mathrm{sm}(S_1)$.

Finally, it is easy to see that $\Box^-(S_2) \subseteq \Box^-(S_1)$ (therefore, $\mathrm{nm}(S_2) \leqslant \mathrm{nm}(S_1)$), $\Box^+(S_2) \subseteq \Box^+(S_1)$ and also for any l: $\Box_l^-(S_2) \subseteq \Box_l^-(S_1)$ and $\Box_l^+(S_2) \subseteq \Box_l^+(S_1)$. What is more, formula numbering remains unchanged and no formulas of the form $\Box_l^{\ominus i}F$ or $\Box_l^{*}F$ can disappear from the antecedent after application of logical rule, but the new ones can be formed[6]. Thus $\mathrm{P_{ref}}(m, S_2) \supseteq \mathrm{P_{ref}}(m, S_1)$ for any m and also $\mathrm{P_{ref}}(l, m, S_2) \supseteq \mathrm{P_{ref}}(l, m, S_1)$ for any m and l. In the case of application of simplification rule, it is easy to see that two main formulas differ only in numbering and the one with the smaller number is part of the premise too, so in that case $\mathrm{P_{ref}}(m, S_2) = \mathrm{P_{ref}}(m, S_1)$ and $\mathrm{P_{ref}}(l, m, S_2) = \mathrm{P_{ref}}(l, m, S_1)$.

Now if some $\Delta = \{F\} \cup \Delta_1 \in \mathrm{Pow}(S_2)$, then from the fact that $F \in \Box_a^+(S_2)$ follows that $F \in \Box_a^+(S_1)$ and from $\Delta_1 \subseteq \Box_a^-(S_2)$ follows that $\Delta_1 \subseteq \Box_a^-(S_1)$. What is more, if the outmost occurrence of $\Box_a$ in F is not marked in S_2 then it is not marked in S_1 too and $\Delta \in \mathrm{Pow}(S_1)$. Contrary, if the outmost occurrence of $\Box_l$ in F is marked with m_1 in S_2, then it is marked with m_1 in S_1 also and if $\Delta_1 \nsubseteq \mathrm{P_{ref}}(a, m_1, S_2)$, then $\Delta_1 \nsubseteq \mathrm{P_{ref}}(a, m_1, S_1)$. So in this case, $\Delta \in \mathrm{Pow}(S_1)$ too. From this it can be concluded that $\mathrm{Pow}(S_2) \subseteq \mathrm{Pow}(S_1)$ and thus $\mathrm{p}(S_2) \leqslant \mathrm{p}(S_1)$.

The reasoning for $\mathrm{c}(S_2) \leqslant \mathrm{c}(S_1)$ is analogous.

2. In this case it is enough to notice that if the rule $(\Box_l^{\ominus i} \to)$ is applied then the index $\ominus i$ is starred everywhere in the premise. Because new index can not be formed in derivation search tree and stars of other starred indexes are not removed, it must be concluded that $\mathrm{sm}(S_2) = \mathrm{sm}(S_1) - 1$ and thus $\mathrm{sm}(S_2) < \mathrm{sm}(S_1)$.

   The proof of $\mathrm{c}(S_2) \leqslant \mathrm{c}(S_1)$, $\mathrm{p}(S_2) \leqslant \mathrm{p}(S_1)$ and $\mathrm{nm}(S_2) \leqslant \mathrm{nm}(S_1)$ is analogous to the previous case.

[5] For example, the starless modality decreases after application of $(\to \wedge)$ to $\to (\neg\Box_2^{\ominus 1}p) \wedge (\neg\Box_1^{\ominus 2}q)$.
[6] For example, as a result of applying $(\wedge \to)$ to $p \wedge \Box_c^{\ominus 1}q \to$.

3. Let's say that the rule $(\rightarrow \Box^{c,i})$ is applied and in the premise all the occurrences of $\Box_c^i$ are marked with m. It is not necessarily true that $\Box^-(S_2) \subseteq \Box^-(S_1)$, but it is possible to claim that for any formula $G \in \Box^-(S_2)$ there is formula $F \in \Box^-(S_1)$ such that $G = F^{i \leftarrow m}$. In this proof the fact is denoted as $\Box^-(S_2) \sqsubseteq \Box^-(S_1)$ and from this immediately follows that $\mathrm{nm}(S_2) \leqslant \mathrm{nm}(S_1)$.

   Similarly it is possible to say that $\Box_c^+(S_2) \sqsubseteq \Box_c^+(S_1)$.

   Now let's say that some set of formulas $\Delta = \{F^{i \leftarrow m}\} \cup \Delta_1^{i \leftarrow m} \in \mathrm{CPow}(S_2)$. From the fact that $F^{i \leftarrow m} \in \Box_c^+(S_2)$ follows that $F \in \Box_c^+(S_1)$ and from the fact that $\Delta_1^{i \leftarrow m} \subseteq \Box^-(S_2)$ follows that $\Delta_1 \subseteq \Box^-(S_1)$. Finally if the outmost occurrence of $\Box_c$ in $F^{i \leftarrow m}$ is not marked, then the outmost occurrence of $\Box_c$ in F is also not marked and thus $\{F\} \cup \Delta_1 \in \mathrm{CPow}(S_1)$. In the other case if the outmost occurrence of $\Box_c$ in $F^{i \leftarrow m}$ is marked by m_2, then either the outmost occurrence of $\Box_c$ in F is not marked and consequently $\{F\} \cup \Delta_1 \in \mathrm{CPow}(S_1)$ or it is marked by $m_1 \leqslant m_2$. Now, all the formulas of the type $\Box_l^* G$ from the conclusion of the application of central agent rule are preserved with the same numbers in the premise except the mark of modality $\Box_c^i$ is changed. Thus it can be said that $\mathrm{P}_{\mathrm{ref}}(m_2, S_2) \sqsupseteq \mathrm{P}_{\mathrm{ref}}(m_2, S_1)$. What is more, from the fact that $m_1 \leqslant m_2$ and from the definition of $\mathrm{P}_{\mathrm{ref}}(m, S)$ follows that $\mathrm{P}_{\mathrm{ref}}(m_2, S_1) \supseteq \mathrm{P}_{\mathrm{ref}}(m_1, S_1)$. Therefore, because $\Delta_1^{i \leftarrow m} \nsubseteq \mathrm{P}_{\mathrm{ref}}(m_2, S_2)$ it can be said that $\Delta_1 \nsubseteq \mathrm{P}_{\mathrm{ref}}(m_2, S_1)$ and consequently $\Delta_1 \nsubseteq \mathrm{P}_{\mathrm{ref}}(m_1, S_1)$. Once again it must be concluded that $\{F\} \cup \Delta_1 \in \mathrm{CPow}(S_1)$.

   Here it is shown that $\mathrm{c}(S_2) \leqslant \mathrm{c}(S_1)$. To remove the "or equal" part, set Γ must be found, that is part of $\mathrm{CPow}(S_1)$, but $\Gamma^{i \leftarrow m}$ is not part of $\mathrm{CPow}(S_2)$. Let's say that the central agent rule is applied to the formula $\Box_c^{i(m_1)} F$. If $(m_1) = \varnothing$, then according to the definition, the set $\{\Box_c^i F\} \in \mathrm{CPow}(S_1)$. If $\Box_c^{i(m)} F$ is not part of $\Box_c^+(S_2)$, then $\{\Box_l F\} \notin \mathrm{CPow}(S_2)$. Otherwise, the set $\{\Box_c^{i(m)} F\} \notin \mathrm{Pow}(S_2)$, because $\varnothing \subseteq \mathrm{P}_{\mathrm{ref}}(m, S_2)$.

   If $m_1 \neq \varnothing$, then $m_1 < m$. Now if the central agent rule is applied, then the condition has to be met meaning that there is formula $[\Box_l^{*i} G]^n$ in S_1 and $m_1 < n$. At the same time (remember Lemma 3.4.7) $n < m$. Because simplification rule cannot be applied to $[\Box_l^{*i} G]^n$ in S_1, $\{\Box_l^{*i} G\} \nsubseteq \mathrm{P}_{\mathrm{ref}}(m_1, S_1)$, however $\{\Box_l^{*i} G^{i \leftarrow m}\} \subseteq \mathrm{P}_{\mathrm{ref}}(m, S_2)$. Thus it can be concluded that $\{\Box_c^{i(m_1)} F, \Box_l^{*i} G\} \in \mathrm{CPow}(S_1)$, but $\{\Box_c^{i(m)} F, \Box_l^{*i} G^{i \leftarrow m}\} \notin \mathrm{CPow}(S_2)$ and that $\mathrm{c}(S_2) < \mathrm{c}(S_1)$.

Now, it is easy to show that for any agent l it is true that $\Box_l^+(S_2) \sqsubseteq \Box_l^+(S_1)$ and $\Box_l^-(S_2) \sqsubseteq \Box_l^-(S_1)$. What is more, only marks of central agent modality is altered and marks of modalities of other agents are not changed. Moreover, every formula of the form $[\Box_l^{*j} H]^n$, that is part of S_1 is also part of S_2 with the same number. Therefore, $\mathrm{P_{ref}}(l, m, S_2) \sqsupseteq \mathrm{P_{ref}}(l, m, S_1)$ for any agent l and any mark m. Thus, by analogous reasoning as in the part of logical rules, it can be concluded that $\mathrm{Pow}(S_2) \sqsubseteq \mathrm{Pow}(S_1)$ and therefore $\mathrm{p}(S_2) \leqslant \mathrm{p}(S_1)$.

4. Once again, it is not hard to see that $\mathrm{nm}(S_2) \leqslant \mathrm{nm}(S_1)$. Let's say that a-transitivity rule is applied to formula $\Box_a^{i(m)} F$ and after the application m is changed to m_1. Once again, it is obvious, that for any agent l it is true that $\Box_l^+(S_2) \sqsubseteq \Box_l^+(S_1)$ and $\Box_l^-(S_2) \sqsubseteq \Box_l^-(S_1)$. If $m \neq \varnothing$, then only occurrences of $\Box_a$ and possibly $\Box_c$ can be marked. Thus all the occurrences of $\Box_{a'}$ for any $a' \neq a$ (and $a' \neq c$) are not marked in both S_1 and S_2. Therefore, for any $F \in \Box_{a'}^+(S_1)$, $a' \neq a$ it is true that if set $\{F^{i \leftarrow m_1}\} \cup \Delta_1^{i \leftarrow m_1} \in \mathrm{Pow}(S_2)$, then $\{F\} \cup \Delta_1 \in \mathrm{Pow}(S_1)$. Moreover, $\mathrm{P_{ref}}(a, m', S_2) \sqsupseteq \mathrm{P_{ref}}(a, m', S_1)$ for any m'. From this fact by the analogous reasoning to the central agent rule part follows that for any $F \in \Box_a^+(S_1)$ it is true that if set $\{F^{i \leftarrow m_1}\} \cup \Delta_1^{i \leftarrow m_1} \in \mathrm{Pow}(S_2)$, then $\{F\} \cup \Delta_1 \in \mathrm{Pow}(S_1)$. In an analogous way to the central agent rule part it is possible to find a set $\Delta \in \mathrm{Pow}(S_1)$ such that $\Delta^{i \leftarrow m_1} \notin \mathrm{Pow}(S_2)$. Therefore, $\mathrm{p}(S_1) < \mathrm{p}(S_2)$.

   If however $m = \varnothing$, then if there are no marked modalities in S_1, except possibly $\Box_c$, then the case is analogous to the one where $m \neq \varnothing$. Otherwise, let $\Box_{a'}^{i'(m')} G$ be part of $\Box^+(S_1)$. Because the formula is marked, there obviously exists S_2' below S_1 in the same branch, which is a premise of application of a'-transitivity rule to $\Box_{a'}^{i'(m'')} G$, and there are no applications of a_1-transitivity rule, $a_1 \neq a'$, between S_1 and S_2'. Let the conclusion of such application be S_1'. Because $\Box_{a'}^{i'(m'')} G$ is part of the succedent of S_1' and it is also subformula of some formula of S_1, then its superformula must be part of the antecedent of S_1'. Moreover, for such formula to be part of S_2' it must be of the form $\Box_{a'}^{*j} H$. Let $\Box_{a'}^{*i''} G'$ be the longest such formula. There is no superformula of $\Box_{a'}^{*i''} G'$ in S_2'. What is more, this formula is part of antecedent of S_1 and it has no superformula in S_1 also. Therefore, $\Box_{a'}^{*i''} G' \in \Box^-(S_1)$, but

$\Box_{a'}^{*i''} G' \notin \Box^-(S_2)$. It is easy to see, that $\Box^-(S_2) \sqsubseteq \Box^-(S_1)$. Therefore, $\Box^-(S_2) \sqsubset \Box^-(S_1)$ and $\mathrm{nm}(S_2) < \mathrm{nm}(S_1)$.

$\Box$

The next task is to show, that neither measure can decrease infinitely.

Lemma 3.4.13. *In any derivation search tree of $G^{*}S4_n^c$ if for some labelled sequent S:*

1. $\mathrm{l}(S) = 0$ *then it is not possible to apply any logical or simplification rule to S.*
2. $\mathrm{sm}(S) = 0$ *then it is not possible to apply reflexivity rule to S.*
3. $\mathrm{c}(S) = 0$ *then it is not possible to apply central agent rule to S.*
4. $\mathrm{p}(S) = 0$ *then it is not possible to apply any transitivity rule to S.*

Finally, $\mathrm{nm}(S) \geqslant 0$.

Proof. If $\mathrm{l}(S) = 0$ then by the definition there are no logical operators in S and thus neither logical nor simplification rule can be applied to S.

If $\mathrm{sm}(S) = 0$ then there are no not-starred negative indexes in S but reflexivity rule can only be applied to the formulas of the type $\Box_l^{\ominus i} F$.

Say that $\mathrm{c}(S) = 0$ and central agent rule $(\to \Box^{c,i})$ is applied to formula $\Box_c^{i(m)} F$. If $(m) = \varnothing$ then $\{\Box_c^{i} F\} \in \mathrm{Pow}(S)$ and thus $\mathrm{p}(S) > 0$. Otherwise the condition of central agent rule must be met, meaning that there is formula $[\Box_l^{*j} G]^n$ in antecedent of S such that $m < n$. In that case $\{\Box_l^{*j} G\} \not\subseteq \mathrm{P}_{\mathrm{ref}}(m, S)$, therefore $\{\Box_c^{i(m)} F, \Box_l^{*j} G\} \in \mathrm{Pow}(S)$ and $\mathrm{p}(S) > 0$. Both cases result in contradictions so it must be concluded that central agent rule can only be applied if $\mathrm{c}(S) > 0$ and consequently if $\mathrm{c}(S) = 0$ central agent rule can not be applied to S. The case of $\mathrm{p}(S) = 0$ is analogous.

The fact that $\mathrm{nm}(S) \geqslant 0$ follows immediately from the definition of $\mathrm{nm}(S)$. $\Box$

The direct corollary of Lemmas 3.4.12 and 3.4.13 is the following theorem.

Theorem 3.4.14. *Calculus $G^{*}S4_n^c$ is terminating.*

Proof. The reasoning is the same as in the proof of Theorem 3.1.3. $\Box$

3.4.3 Soundness and Completeness of the Calculus

Now it must be shown that calculus $G^{*}S4_n^c$ is sound and complete. For this, calculus $GS4_n^c$, which is sound and complete, is used. To prove the soundness the admissibility of weakening structural rule must be proved.

Lemma 3.4.15. *Weakening structural rule is admissible in $GS4_n^c$.*

Proof. The proof is analogous to the proof of Lemma 1.4.8. □

Theorem 3.4.16. *Calculus $G^{*}S4_n^c$ is sound. That is for any sequent S if labelled sequent* Lab(S) *is derivable in $G^{*}S4_n^c$ then sequent S is derivable in $GS4_n^c$.*

Proof. To prove this first it must be noticed that logical rules and reflexivity rule of $G^{*}S4_n^c$ are the same as in $GS4_n^c$ except that the rules of $GS4_n^c$ do not contain labels. What is more, after removing all the labels from the application of some succedental rule of $G^{*}S4_n^c$, it can be replaced by application of respective succedental rule and some applications of reflexivity rule of $GS4_n^c$. Next, simplification rule of $G^{*}S4_n^c$ can be replaced by weakening structural rule, which is admissible in $GS4_n^c$. Finally, after deletion of labels from the axioms of $G^{*}S4_n^c$, axioms of $GS4_n^c$ are obtained. Therefore a derivation tree in $G^{*}S4_n^c$ can be easily transformed to the derivation tree in $GS4_n^c$. □

To prove the completeness of $G^{*}S4_n^c$ several intermediate calculi are used. This is done to make the proof more clear by introducing one change to the calculus at a time, therefore each intermediate calculus differs from the previous one in one aspect only.

First of all $G^{*}S4_n^c$-style succedental rules are introduced to $GS4_n^c$.

Definition 3.4.17. *Let $G_1S4_n^c$ be the calculus obtained from $GS4_n^c$ by changing succedental rules to:*

$$\frac{\Gamma_1, \Box_l \Gamma_1 \to F}{\Box_l \Gamma_1, \Gamma_2 \to \Delta, \Box_l F}\,(\to \Box_l)_2 \qquad \frac{\Gamma_1, \Box_* \Gamma_1 \to F}{\Box_* \Gamma_1, \Gamma_2 \to \Delta, \Box_c F}\,(\to \Box^c)_2$$

Lemma 3.4.18. *If a sequent is derivable in $GS4_n^c$, then it is derivable in $G_1S4_n^c$.*[7]

Proof. Let's say that $\mathcal{D}$ is a derivation tree of sequent S in $GS4_n^c$. If there are no applications of succedental rules in $\mathcal{D}$, then $\mathcal{D}$ is already a derivation

[7] Analogously to Footnote 1 in page 77, although $G_1S4_n^c$ is sound, only completeness of the calculus is proved. Soundness is not needed for the completeness proof of the loop-free calculus $G^{*}S4_n^c$. This note applies to all the intermediate calculi.

tree in $G_1S4_n^c$. Otherwise, let's take the lowest such inference and let S_1 be its premise. Say that it is an application of transitivity rule $(\rightarrow \Box_l)$. The case of central agent rule is completely analogous. Now, because of invertibility of reflexivity rule, after such application of rule it is possible to add one application of $(\Box_l \rightarrow)$ for each formula of the form $\Box_l F$ in the antecedent of S_1. Then such application of transitivity rule and recently introduced applications of reflexivity rule can be replaced with single application of $(\rightarrow \Box_l)_2$ to get derivation tree $\mathcal{D}_1$. If $\mathcal{D}_1$ still contains at least one application of succedental rule, then let's continue this change inductively to the lowest application of succedental rule in $\mathcal{D}_1$. Otherwise, $\mathcal{D}_1$ is already a derivation tree in $G_1S4_n^c$. □

Next, a prohibition to apply transitivity rule for central agent is included.

Definition 3.4.19. *Let $G_2S4_n^c$ be the calculus obtained from $G_1S4_n^c$ by replacing the transitivity rule to:*

$$\frac{\Gamma_1, \Box_a\Gamma_1 \rightarrow F}{\Box_a\Gamma_1, \Gamma_2 \rightarrow \Delta, \Box_a F}\ (\rightarrow \Box_a)_2$$

where a is any agent, except the central one.

To prove the completeness, admissibility of weakening is needed.

Lemma 3.4.20. *Weakening structural rule is admissible in $G_2S4_n^c$.*

Proof. The proof is analogous to the proof of Lemma 1.4.8. □

Lemma 3.4.21. *If a sequent is derivable in $G_1S4_n^c$, then it is derivable in $G_2S4_n^c$.*

Proof. Let's say that $\mathcal{D}$ is a derivation tree of sequent S in $G_1S4_n^c$. If there are no applications of c-transitivity rule, then $\mathcal{D}$ is already a derivation tree in $G_2S4_n^c$. Otherwise, let's take the topmost such inference:

$$\cfrac{\cfrac{\mathcal{D}_1}{\Gamma_1, \Box_c\Gamma_1 \rightarrow F}}{\Box_c\Gamma_1, \Gamma_2 \rightarrow \Delta, \Box_c F}\ (\rightarrow \Box_c)_2$$

Note, that because this is the topmost application of $(\rightarrow \Box_c)_2$, derivation tree $\mathcal{D}_1$ is already a derivation tree in $G_2S4_n^c$. Now to remove this application of transitivity rule, let's change it to the application of central agent rule and some applications of weakening structural rules:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma_1, \Box_c\Gamma_1 \to F}}{\Gamma_1, \Box_c\Gamma_1, \Gamma_2^{'}, \Box_{\neq c}\Gamma_2^{'} \to F}}{\Box_c\Gamma_1, \Box_{\neq c}\Gamma_2^{'}, \Gamma_2^{''} \to \Delta, \Box_c F}\,(\to \Box^c)_2$$

By inductively continuing this change, it is possible to remove all the applications of c-transitivity rule. $\Box$

Now, labels are introduced to the calculus.

Definition 3.4.22. *Let $G_3S4_n^c$ be the labelled sequent calculus obtained from $G_2S4_n^c$ by adapting it to labelled sequents. That is, the axiom is changed to $\Gamma, [F_1]^{n_1} \to [F_2]^{n_2}, \Delta$, where $\mathrm{Proj}(F_1) = \mathrm{Proj}(F_2)$, logical rules are changed to logical rules of $G^*S4_n^c$, and reflexivity, transitivity and central agent rules are replaced by:*

Two rules of reflexivity:

$$\dfrac{[F]^n, [\Box_l^{*i}F]^n, \Gamma^{\ominus i*} \to \Delta^{\ominus i*}}{[\Box_l^{\ominus i}F]^n, \Gamma \to \Delta}\,(\Box_l^{\ominus i} \to) \qquad \dfrac{[F]^n, [\Box_l^{*i}F]^n, \Gamma \to \Delta}{[\Box_l^{*i}F]^n, \Gamma \to \Delta}\,(\Box_l^{*i} \to)$$

Transitivity rule:

$$\dfrac{\left\{[\Gamma_1^{i\leftarrow n}]^n, \Box_a^{\lambda}\Gamma_1^{i\leftarrow n} \to [F]^n\right\}^{\neq a:\not *,(\not)}}{\Box_a^{\lambda}\Gamma_1, \Gamma_2 \to \Delta, [\Box_a^{i(m)}F]^{n-1}}\,(\to \Box_a^i)^{\lambda}$$

where $a \neq c$, $\Box_a^{\lambda}\Gamma_1$ consists of formulas of the form $\Box_a^{\ominus j}G$ or $\Box_a^{*j}G$, i is some index or $i = \varnothing$, (m) is some mark or $(m) = \varnothing$.
Central agent rule:

$$\dfrac{[\Gamma^{i\leftarrow n}]^n, \Box_*^{\lambda}\Gamma^{i\leftarrow n} \to [F]^n}{\Box_*^{\lambda}\Gamma_1, \Gamma_2 \to \Delta, [\Box_c^{i(m)}F]^{n-1}}\,(\to \Box^{c,i})^{\lambda}$$

where $\Box_*^{\lambda}\Gamma$ consists of formulas of the form $\Box_l^{\ominus j}G$ or $\Box_l^{*j}G$, i is some index or $i = \varnothing$, (m) is some mark or $(m) = \varnothing$.

Lemma 3.4.23. *If sequent S is derivable in $G_2S4_n^c$ then labelled sequent $\mathrm{Lab}(S)$ is derivable in $G_3S4_n^c$.*

Proof. Note, that because of extra reflexivity rule and different formulation of succedental rules labels do not add any restrictions to the calculus. That is, to get a derivation tree in $G_3S4^c_n$ it is only needed to add appropriate labels to every sequent of derivation tree in $G_2S4^c_n$. The labelling of initial sequent is according to the definition 3.4.4. After that if the conclusion of some inference is a labelled sequent, then labels are introduced to the premise (or premises) as described in the definition of the rules of $G_3S4^c_n$. No application of rule is added or removed from the derivation tree in $G_2S4^c_n$. ∎

$G_3S4^c_n$ and all the following intermediate calculi are labelled. As mentioned earlier, in a derivation search tree of labelled sequent calculus the initial sequent must be obtained by labelling of the sequent that does not contain any labels. This is important for the correctness of some lemmas.

Now, let's introduce next restriction. This one restricts applications of succedental rules. The aim is to prohibit the application of succedental rule to the sequent if some logical or $(\Box_l^{\ominus i} \to)$ rule of $G_3S4^c_n$ can be applied to it. This change is introduced by changing the succedental rules.

Definition 3.4.24. *Let $G_4S4^c_n$ be the labelled sequent calculus obtained from $G_3S4^c_n$ by changing succedental rules to:*

Transitivity

$$\frac{\left\{[\Gamma_1^{i\leftarrow n}]^n, \Box_a^*\Gamma_1^{i\leftarrow n} \to [F]^n\right\}^{\neq a:\not*,(\not)}}{\Box_a^*\Gamma_1, \Box_{\neq a}^*\Gamma_2, \Sigma_1 \to \Sigma_2, \Box_*\Delta, [\Box_a^{i(m)}F]^{n-1}}\ (\to \Box_l^i)$$

where $a \neq c$, Σ_1 and Σ_2 are empty or consist of propositional variables only, i is some index or $i = \varnothing$, (m) is some mark or $(m) = \varnothing$.

Central agent rule:

$$\frac{[\Gamma^{i\leftarrow n}]^n, \Box_*^*\Gamma^{i\leftarrow n} \to [F]^n}{\Box_*^*\Gamma, \Sigma_1 \to \Sigma_2, \Box_*\Delta, [\Box_c^{i(m)}F]^{n-1}}\ (\to \Box^{c,i})$$

The meaning of the notation is the same as in transitivity rule.

To prove the completeness of this calculus, invertibility of logical and reflexivity rules are needed.

Lemma 3.4.25. *Logical, $(\Box_l^{\ominus i} \to)$ and $(\Box_l^{*i} \to)$ rules are invertible in $G_3S4^c_n$.*

Proof. The proof is analogous to the proof of Lemmas 1.4.11 and 1.4.12. ∎

Lemma 3.4.26. *If a labelled sequent is derivable in $G_3S4^c_n$ then it is derivable in $G_4S4^c_n$.*

Proof. This is a direct corollary of Lemma 3.4.25. □

Next the restriction on applications of reflexivity rule is introduced. That is, all the applications of $(\Box_l^{*i} \to)$ rule are removed.

Definition 3.4.27. *Let $G_5S4_n^c$ be the labelled sequent calculus obtained from $G_4S4_n^c$ by removing the rule $(\Box_l^{*i} \to)$.*

To prove the completeness of $G_5S4_n^c$, admissibility of contraction structural rules must be shown.

Lemma 3.4.28. *Contraction structural rules are admissible in $G_5S4_n^c$.*

Proof. The proof is analogous to the proof of Lemma 1.4.13. □

Lemma 3.4.29. *If a labelled sequent is derivable in $G_4S4_n^c$, then it is derivable in $G_5S4_n^c$.*

Proof. For this proof the marks are completely ignored. Moreover, negative indexes are also ignored, except when needed for the proof. Thus it is said that $\Box_l^{*i}F$ and $\Box_l^{*j}G$ are equal, if $\mathrm{Proj}(F) = \mathrm{Proj}(G)$. This is done to simplify the proof, however as indicated later this does not change the correctness of it.

Let's say that $\mathcal{D}$ is a derivation tree of labelled sequent S in $G_4S4_n^c$. Let's denote the number of applications of $(\Box_l^{*i} \to)$ in $\mathcal{D}$ for any l and i by $\mathrm{r}(\mathcal{D})$. If $\mathrm{r}(\mathcal{D}) = 0$, then $\mathcal{D}$ is already a derivation tree in $G_5S4_n^c$. Otherwise, let's take the topmost such application:

$$\frac{S_2 = [F]^n, [\Box_l^{*i}F]^n, \Gamma \to \Delta}{S_1 = [\Box_l^{*i}F]^n, \Gamma \to \Delta}\,(\Box_l^{*i} \to)$$

Initial sequent S does not contain starred modalities. Moreover, all stars from modalities of agent l are removed after application of rule l_1-transitivity rule, where $l_1 \neq l$. Thus, below S_1 there must be sequent $S_1^{'}$, which is a conclusion of application of $(\Box_l^{\ominus i} \to)$ rule to $\Box_l^{\ominus i}F$ and there are no applications of l_1-transitivity rule between S_1 and $S_1^{'}$, except possibly if $l_1 = l$[8]:

$$\begin{array}{c}\dfrac{S_2 = [F]^n, [\Box_l^{*i}F]^n, \Gamma \to \Delta}{S_1 = [\Box_l^{*i}F]^n, \Gamma \to \Delta}\,(\Box_l^{*i} \to)\\ \cdots \\ \dfrac{S_2^{'} = [F]^{n_1}, [\Box_l^{*i}F]^{n_1}, \Gamma^{'} \to \Delta^{'}}{S_1^{'} = [\Box_l^{\ominus i}F]^{n_1}, \Gamma^{'} \to \Delta^{'}}\,(\Box_l^{\ominus i} \to)\end{array}$$

[8]In fact, F in S_1 and F in $S_1^{'}$ can differ in marks and some negative indexes in the latter could be starred in the former. However this difference is ignored, because the inside of the formula F is changed by other rules, that are not eliminated after elimination of inappropriate application of reflexivity rule.

Let $\mathcal{D}_1$ be a path of the derivation from $S_2^{'}$ to S_1. If there are no applications of succedental rules in $\mathcal{D}_1$, then let's apply the rule $(c \rightarrow)$ to $[F]^{n_1}$ in $S_2^{'}$ and then let's apply all the rules of $\mathcal{D}_1$ to the premise in the same order as they appear in $\mathcal{D}_1$. This results in sequent S_3:

$$\cfrac{\cfrac{\begin{array}{c} S_3 = [F]^{n_1}, [\Box_l^{*i} F]^n, \Gamma \rightarrow \Delta \\ \dots \\ [F]^{n_1}, [F]^{n_1}, [\Box_l^{*i} F]^{n_1}, \Gamma^{'} \rightarrow \Delta^{'} \end{array}}{S_2^{'} = [F]^{n_1}, [\Box_l^{*i} F]^{n_1}, \Gamma^{'} \rightarrow \Delta^{'}} \, (c \rightarrow)}{S_1^{'} = [\Box_l^{\ominus i} F]^{n_1}, \Gamma^{'} \rightarrow \Delta^{'}} \, (\Box_l^{\ominus i} \rightarrow)$$

Notice that S_2 differs from S_3 only with respect to the number of the formula F. Therefore, after making this change, all the appropriate numbers and marks in sequents of derivation above S_3 have to be changed. The result of these changes is derivation tree $\mathcal{D}_2$ and $\mathrm{r}(\mathcal{D}_2) < \mathrm{r}(\mathcal{D})$.

Otherwise if there is at least one application of succedental rule in $\mathcal{D}_1$, let's take the one closest to S_1. Let $\mathcal{D}_3$ be a path of $\mathcal{D}_1$ from the premise of such application to S_1. Formula $[\Box_l^{*i} F]^{n_1}$ is definitely part of the antecedent of conclusion of such application, therefore the antecedent of the premise contains $[F]^{n_2}$. Let's apply $(c \rightarrow)$ to $[F]^{n_2}$ in the premise and afterwards let's apply all the rules of $\mathcal{D}_3$ in the same order. Only the case of l-transitivity rule is displayed, because the case of central agent rule is completely analogous.

$$\cfrac{\cfrac{\begin{array}{c} S_4 = [F]^{n_2}, [\Box_l^{*i} F]^n, \Gamma \rightarrow \Delta \\ \dots \\ [F]^{n_2}, [F]^{n_2}, [\Box_l^{*i} F]^{n_1}, [\Gamma_1^{''}]^{n_2}, \Box_l^{*} \Gamma_1^{''} \rightarrow [G]^{n_2} \end{array}}{[F]^{n_2}, [\Box_l^{*i} F]^{n_1}, [\Gamma_1^{''}]^{n_2}, \Box_l^{*} \Gamma_1^{''} \rightarrow [G]^{n_2}} \, (c \rightarrow)}{[\Box_l^{*i} F]^{n_1}, \Box_l^{*} \Gamma_1^{''}, \Box_{\neq l}^{*} \Gamma_2^{''}, \Sigma_1 \rightarrow \Sigma_2, \Box_{*} \Delta^{''}, [\Box_l^{j} G]^{n_3}} \, (\rightarrow \Box_l^{j})$$

$$\cfrac{\begin{array}{c} \dots \\ S_2^{'} = [F]^{n_1}, [\Box_l^{*i} F]^{n_1}, \Gamma^{'} \rightarrow \Delta^{'} \end{array}}{S_1^{'} = [\Box_l^{\ominus i} F]^{n_1}, \Gamma^{'} \rightarrow \Delta^{'}} \, (\Box_l^{\ominus i} \rightarrow)$$

Once again S_2 differs from S_4 only with respect to the number of the formula F and all the appropriate numbers and marks in sequents of derivation above S_4 must be changed. The result of such changes is derivation tree $\mathcal{D}_4$ and $\mathrm{r}(\mathcal{D}_4) < \mathrm{r}(\mathcal{D})$.

Thus by applying these changes inductively it is possible to completely eliminate applications of $(\Box_l^{*i} \rightarrow)$ rule for any l and i. $\Box$

Now, let's insert simplification rule to the calculus.

Definition 3.4.30. *Let $G_6S4_n^c$ be the labelled sequent calculus obtained from $G_5S4_n^c$ by adding simplification rule:*

$$\frac{[\Box_l^{*i}F]^{n_1}, \Gamma \to \Delta}{[\Box_l^{*i}F]^{n_1}, [\Box_l^{*i}F]^{n_2}, \Gamma \to \Delta}\ (\Box_l^{n_1,n_2} \to)$$

where $n_1 < n_2$. If possible, simplification rule must be applied first.

Once again, admissibility of contraction structural rules is needed.

Lemma 3.4.31. *Contraction structural rules are admissible in $G_6S4_n^c$.*

Proof. The proof is analogous to the proof of Lemma 1.4.13. □

Lemma 3.4.32. *If a labelled sequent is derivable in $G_5S4_n^c$ then it is derivable in $G_6S4_n^c$.*

Proof. Let's say that $\mathcal{D}$ is a derivation tree in $G_5S4_n^c$. Let's find the topmost sequent of the form $[\Box_l^{*i}F]^{n_1}, [\Box_l^{*i}F]^{n_2}, \Gamma \to \Delta$, where $n_1 < n_2$, in $\mathcal{D}$. If there is no such sequent, then $\mathcal{D}$ is already a derivation tree in $G_6S4_n^c$. Otherwise, let's apply simplification rule to the sequent. Now, it is possible to apply contraction structural rule to the premise of newly introduced application of simplification rule:

$$\cfrac{\cfrac{[\Box_l^{*i}F]^{n_1}, [\Box_l^{*i}F]^{n_1}, \Gamma \to \Delta}{[\Box_l^{*i}F]^{n_1}, \Gamma \to \Delta}\ (c \to)}{[\Box_l^{*i}F]^{n_1}, [\Box_l^{*i}F]^{n_2}, \Gamma \to \Delta}\ (\Box_l^{n_1,n_2} \to)$$

This derivation can be continued as in $\mathcal{D}$, except numbers and marks of sequents above the premise of application of $(c \to)$ must be altered.

This procedure can be repeated inductively until a derivation tree in $G_6S4_n^c$ is obtained. □

It should be noticed, that any derivation in $G_6S4_n^c$ and following intermediate calculi follows the conditions set out in Lemma 3.4.7.

Finally the other restriction on applications of succedental rules is inserted.

Definition 3.4.33. *Let $G_7S4_n^c$ be the labelled sequent calculus obtained from $G_6S4_n^c$ by adding the following restrictions:*

1. *Transitivity rule with main formula $[\Box_a^{i(m)}F]^n$ can be applied to the sequent $\Box_a^*\Gamma_1, \Box_{\neq a}^*\Gamma_2, \Sigma_1 \to \Sigma_2, \Box_*\Delta, [\Box_a^{i(m)}F]^n$ if either $(m) = \varnothing$ or $\Box_a^*\Gamma_1$ contains at least one formula of the form $[\Box_a^{*j}C]^{n_1}$ where $m < n_1 \leqslant n$.*

2. *Central agent rule with main formula $[\Box_c^{i(m)} F]^n$ can be applied to the sequent $\Box_*^* \Gamma, \Sigma_1 \to \Sigma_2, \Box_* \Delta, [\Box_c^{i(m)} F]^n$ if either $(m) = \varnothing$ or $\Box_*^* \Gamma$ contains at least one formula of the form $[\Box_l^{*j} C]^{n_1}$ for some l, where $m < n_1 \leqslant n$.*

Lemma 3.4.34. *If a labelled sequent is derivable in $G_6 S4_n^c$ then it is derivable in $G_7 S4_n^c$.*

Proof. Our task is to remove all the applications of succedental rules that do not follow the newly introduced restriction. That is, a-transitivity rule is applied to formula marked by m and in the antecedent all the formulas of the form $\Box_a^{*j} G$ are numbered lower than or equal to m or central agent rule is applied to formula marked by m and in the antecedent all the formulas of the form $\Box_l^{*j} G$ for any l are numbered lower than or equal to m. Such applications are called inappropriate and are denoted as $(\to \Box_a^i)_{\text{inappropriate}}$ and $(\to \Box^{c,i})_{\text{inappropriate}}$.

Say, that $\mathcal{D}$ is a derivation tree of labelled sequent S in $G_6 S4_n^c$. Let $\mathrm{s}(\mathcal{D})$ be the number of inappropriate applications of succedental rules. If $\mathrm{s}(\mathcal{D}) = 0$, then $\mathcal{D}$ is already a derivation tree in $G_7 S4_n^c$. Otherwise, let's take the lowest (closest to S) inappropriate application of succedental rule. Let S_1 be the conclusion of such inference.

First, let's analyse the case of a-transitivity rule. The main formula of the application is marked, but all the marks of modalities of agent a are removed after the application of a_1-transitivity rule, where $a_1 \neq a$. Therefore, below S_1 there must be a sequent, which is a conclusion of application of $(\to \Box_a^i)$ rule to the same formula and there are no applications of a_1-transitivity rule between S_1 and this sequent, such that $a_1 \neq a$. Let $S_1^{'}$ be such sequent, that is closest to S_1:

$$\dfrac{S_2 = \left\{[\Gamma_1^{i \leftarrow n}]^n, \Box_a^* \Gamma_1^{i \leftarrow n} \to [F]^n\right\}^{\neq a: \not *, (\not\,)}}{S_1 = \Box_a^* \Gamma_1, \Box_{\neq a}^* \Gamma_2, \Sigma_1 \to \Sigma_2, \Box_* \Delta, [\Box_a^{i(m)} F]^{n-1}} \; (\to \Box_a^i)_{\text{inappropriate}}$$

$$\dots$$

$$\dfrac{S_2^{'} = \left\{[(\Gamma_1^{'})^{i \leftarrow m}]^m, (\Box_a^* \Gamma_1^{'})^{i \leftarrow m} \to [F]^m\right\}^{\neq a: \not *, (\not\,)}}{S_1^{'} = \Box_a^* \Gamma_1^{'}, \Box_{\neq a}^* \Gamma_2^{'}, \Sigma_1^{'} \to \Sigma_2^{'}, \Box_* \Delta^{'}, [\Box_a^{i(m_1)} F]^{m-1}} \; (\to \Box_a^i)$$

Now because of the calculus, it is obvious that $(\Box_a^* \Gamma_1^{'})^{i \leftarrow m} \subseteq \Box_a^* \Gamma_1$. What is more, as the presumption states, there are no formulas in $\Box_a^* \Gamma_1$ with numbers greater than m. Therefore all the formulas that are part of $\Box_a^* \Gamma_1$, but do not belong to $(\Box_a^* \Gamma_1^{'})^{i \leftarrow m}$ are numbered m. Let's say that $\Box_a^* \Gamma_1 = [\Box_a^{*i_1} G_1]^m, \dots, [\Box_a^{*i_k} G_k]^m, (\Box_a^* \Gamma^{'})^{i \leftarrow m}$.

Now let's follow a path of $\mathcal{D}$ from $S_2^{'}$ to S_2. Let's start from $S_2^{'}$. If there is an application of l-transitivity or central agent rule before S_1, then let $S_3^{'}$ be the conclusion of the first such application. Otherwise let $S_3^{'} = S_1$. Neither logical, nor reflexivity, nor simplification rules can be applied to $S_3^{'}$. Moreover, $[\Box_a^{*i_j} G_j]^m, j \in [1, k]$ are part of antecedent of $S_3^{'}$, because after the application of succedental rule to $S_3^{'}$ all the new formulas are going to be numbered $m + 1$. Therefore, there is an application of reflexivity rule between $S_2^{'}$ and $S_3^{'}$ for each $[\Box_a^{\ominus i_j} G_j]^m$. To sum up, $S_3^{'}$ is obtained from $S_2^{'}$ by applying all the possible logical and simplification rules to formula $[F]^m$, formulas $[G_j]^m, j \in [1, k]$, formulas from $[(\Gamma_1^{'})^{i \leftarrow m}]^m$ and their subformulas.

Now, let's follow derivation tree $\mathcal{D}$ above S_2. Let $\mathcal{S}$ be the set of different labelled sequents and for every $S_j^* \in \mathcal{S}$, S_j^* is the first sequent in a branch above S_2 that is either an axiom or a conclusion of application of succedental rule. Once again, every sequent of $\mathcal{S}$ is obtained from S_2 by applying all the possible logical and simplification rules to formula $[F]^n$, formulas from $[\Gamma_1^{i \leftarrow n}]^n$ (which consists of formulas $[G_j^{i \leftarrow n}]^n, j \in [1, k]$ and formulas from $[(\Gamma_1^{'})^{i \leftarrow n}]^n$) and their subformulas. Therefore, there is a sequent $S_3 \in \mathcal{S}$, such that $\mathrm{Proj}(S_3) = \mathrm{Proj}(S_3^{'})$. This means that all the rules of the derivation tree $\mathcal{D}$ above S_3 can be applied straight to $S_3^{'}$ thus eliminating at least one inappropriate application of transitivity rule and getting derivation tree $\mathcal{D}_1$.

Now, because numbers and marks are different in S_3 and $S_3^{'}$, also numbers and marks in $\mathcal{D}_1$ above $S_3^{'}$ is going to differ from those of $\mathcal{D}$ above S_3. Because of that it must be shown that there are no inappropriate applications of succedental rule in $\mathcal{D}_1$ that are not part of $\mathcal{D}$. To demonstrate that let's compare the marks and numbers in $\mathcal{D}$ and $\mathcal{D}_1$. Having in mind Lemma 3.4.7 the following can be noticed:

- Every formula numbered $n^{'} > m$ in subderivation of $\mathcal{D}_1$ above $S_3^{'}$ (let's call it $\mathcal{D}_3$), is numbered $n^{'} + n - m$ in subderivation of $\mathcal{D}$ above S_3 (let's call it $\mathcal{D}_2$); every formula numbered $n^{'} \leqslant m$ in $\mathcal{D}_3$, is numbered $n^{'}$ in $\mathcal{D}_2$. This is because $\mathcal{D}$ below $S_3^{'}$ is the same as $\mathcal{D}_1$ below $S_3^{'}$. But in derivation $\mathcal{D}_1$ above $S_3^{'}$ all formulas get new numbers in the same order as in $\mathcal{D}$ above S_3, except that in the latter they are larger because of the path of derivation from $S_3^{'}$ to S_3 that is missed out in $\mathcal{D}_1$.

- The contrary is also correct: every formula numbered $n^{'} > n$ in $\mathcal{D}_2$, is numbered $n^{'} - n + m$ in $\mathcal{D}_3$; every formula numbered $n^{'} \leqslant m$ in $\mathcal{D}_2$, is numbered $n^{'}$ in $\mathcal{D}_3$.

- If a formula of the form $\Box_a^{*j}G$ numbered $n^{'}, m < n^{'} \leqslant n$ appears in $\mathcal{D}$ then simplification rule is applied next, that makes the formula disappear from further derivation. Thus no such formulas appear in antecedent of conclusions of succedental rule. This is due to the properties of $\Box_a^{*}\Gamma_1$ and $(\Box_a^{*}\Gamma_1^{'})^{i\leftarrow m}$, that were discussed earlier. Therefore, S_2 (and the derivation above) does not contain formulas numbered $n^{'}$, $m < n^{'} \leqslant n$, and these formulas are not part of $\mathcal{D}_3$.

- Every occurrence of $\Box_l^j$ in $\mathcal{D}_3$ marked with $m^{'} > m$ is marked with $m^{'} + n - m$ in $\mathcal{D}_2$. Every occurrence of $\Box_l^j$ in $\mathcal{D}_3$ marked with $m^{'} \leqslant m$ is marked with $m^{'}$ in $\mathcal{D}_2$.

- Some occurrences of $\Box_a^j$ that are marked in $\mathcal{D}_2$ are not marked in $\mathcal{D}_3$. This is the case, if there is an application of $(\to \Box_a^j)$ rule between $S_2^{'}$ and S_1, $j \neq \varnothing$ and the outermost modality of the main formula of the inference is not marked.

Having this in mind, it is not hard to argue that for any inappropriate application of succedental rule in $\mathcal{D}_3$, the respective application of succedental rule in $\mathcal{D}_2$ is also inappropriate. Thus $\mathrm{s}(\mathcal{D}_1) < \mathrm{s}(\mathcal{D})$.

The case of central agent rule is completely analogous.

By inductively applying this elimination to $\mathcal{D}_1$ it is possible to get a derivation tree in $G_7S4_n^c$. $\Box$

Now the only difference between $G_7S4_n^c$ and $G^*S4_n^c$ is the axiom. In the former it is $\Gamma, [F_1]^{n_1} \to [F_2]^{n_2}, \Delta$, where $\mathrm{Proj}(F_1) = \mathrm{Proj}(F_2)$, and in the latter it is required that $n_1 = n_2$. However, it is easy to show that those too calculi are equivalent. Once again, only one part of the equivalence is mentioned. The other one is obvious.

Lemma 3.4.35. *If an indexed sequent is derivable in $G_7S4_n^c$ then it is derivable in $G^*S4_n^c$.*

Proof. Let $\mathcal{D}$ be a derivation tree in $G_7S4_n^c$. Let's say that in $\mathcal{D}$ there is an axiom $S_1 = \Gamma, [F_1]^{n_1} \to [F_2]^{n_2}, \Delta$ and $n_1 \neq n_2$. It is easy to show that any derivation search tree in $G_7S4_n^c$ follows the conditions set out in Lemma 3.4.7. Therefore, the only possible situation is that $\mathrm{Proj}(F_1) = \mathrm{Proj}(F_2) = \Box_l G$. Moreover, for the same reason $F_1 = \Box_l^{*j} G_1$, where $\mathrm{Proj}(G_1) = G$. Let's say that $F_2 = \Box_l^{i(m)} G_2$, where $\mathrm{Proj}(G_2) = G$, i is some index or $i = \varnothing$, m is some mark or $m = \varnothing$.

Let's apply logical and reflexivity rules of $G_7S4_n^c$ to S_1 and its premises until no such rule can be applied. The result of such process is sequent $S_2 = \Box_l^*\Gamma_1, \Box_{\neq l}^*\Gamma_2, [F_1]^{n_1}, \Sigma_1 \to \Sigma_2, [F_2]^{n_2}, \Box_*\Delta_1$. If it is possible to apply succedental rule to the formula F_2 in S_2 then the premise of such application is going to be sequent $S_3 = \Gamma_3, [G_1]^{n_2+1} \to [G_2]^{n_2+1}$ and $\mathrm{Proj}(G_1) = \mathrm{Proj}(G_2)$. S_3 is axiom of both $G_7S4_n^c$ and $G^*S4_n^c$.

Otherwise, if succedental rule cannot be applied to F_2 in S_2, then $m \neq \varnothing$. If $l \neq c$, then there is no such formula of the form $[\Box_l^{*j'} H]^{n'}$ in $\Box_l^*\Gamma_1, [F_1]^{n_1}$ such that $m < n'$. Therefore, $n_1 \leqslant m$. Moreover, below S_1 there is an application of $(\to \Box_l^i)$ to F_3 such that $\mathrm{Proj}(F_3) = \mathrm{Proj}(F_2)$. Let's denote the premise of such application S_4 and the conclusion S_4'. If $n_1 < m$, then in the antecedent of S_4' there is a formula $[F_4]^{n_1}$ such that $\mathrm{Proj}(F_4) = \mathrm{Proj}(F_1)$. Therefore S_4 is of the form $\Gamma_4, [G_4]^{n_3} \to [G_3]^{n_3}$, where $\mathrm{Proj}(G_4) = \mathrm{Proj}(G_3)$. Thus S_4 is an axiom of both $G_7S4_n^c$ and $G^*S4_n^c$, and the derivation above it is not needed. If $n_1 = m$, then between S_4 and S_2 there is an application of the rule $(\Box_l^{\ominus j'} \to)$ to the formula F_5 such that $\mathrm{Proj}(F_5) = \mathrm{Proj}(F_1)$. Let's say that the premise of such application is S_5. Because $n_1 = m$, there are no applications of succedental rules between S_4 and S_5. Now let's eliminate all the applications of logical and reflexivity rules to formula G_3 and its subformulas from subderivation between S_4 and S_5. This results in $S_6 = \Gamma_5, [G_5]^{n_1} \to [G_3]^{n_1}, \Delta$, where $\mathrm{Proj}(G_5) = \mathrm{Proj}(G_3)$. Once again, it is obvious that S_6 is axiom in both $G_7S4_n^c$ and $G^*S4_n^c$ and the derivation above it should be omitted.

The case, when $l = c$ is completely analogous.

A derivation tree in $G^*S4_n^c$ is obtained by applying this change to every axiom of $\mathcal{D}$, where $n_1 \neq n_2$. $\Box$

From this the completeness of $G^*S4_n^c$ follows immediately.

Theorem 3.4.36. *Calculus $G^*S4_n^c$ is complete. That is, if sequent S is derivable in $GS4_n^c$ then sequent* $\mathrm{Lab}(S)$ *is derivable in $G^*S4_n^c$.*

Proof. A direct consequence of Lemmas 3.4.18, 3.4.21, 3.4.23, 3.4.26, 3.4.29, 3.4.32, 3.4.34 and 3.4.35. $\Box$

3.5 Logics *K4* and $K4_n^c$

As mentioned in [4], the terminating calculus for logic *K4* can be derived using similar ideas as in calculus G^*S4. The only difference between logics of *K4* and *S4* is that reflexivity axiom is not part of *HK4* and therefore reflexivity rule is not part of *GK4*. Because of this, the negative indexation of modalities is not needed.

Definition 3.5.1. *A sequent is called labelled for K4, if every positive occurrence of $\Box$ that is in the scope of negative occurrence of $\Box$ is indexed with integer and all the formulas are numbered.*

Definition 3.5.2. *Labelling for K4 of sequent S is denoted* $\mathrm{Lab}_{K4}(S)$ *and labelled for K4 sequent* $\mathrm{Lab}_{K4}(S)$ *is obtained from S by (1) indexing all the positive occurrences of $\Box$ that are in the scope of negative occurrence of $\Box$ with different natural numbers, and (2) attaching number* 1 *to every formula. No marks are needed in* $\mathrm{Lab}_{K4^c_n}(S)$.

Except these differences, the terminating calculus for *K4* is the same as $G^{*}S4$ with a little change to the restriction of application of transitivity rule.

Definition 3.5.3. *The labelled sequent calculus without loops for logic K4 ($G^{*}K4$) consists of axiom* $\Gamma, [F_1]^n \to [F_2]^n, \Delta$, *where* $\mathrm{Proj}(F_1) = \mathrm{Proj}(F_2)$, *the same logical rules and simplification rule as $G^{*}S4$ and transitivity rule:*

$$\frac{[\Gamma^{i\leftarrow n}]^n, \Box\Gamma^{i\leftarrow n} \to [F]^n}{\Box\Gamma, \Sigma_1 \to \Sigma_2, \Box\Delta, [\Box^{i(m)}F]^{n-1}}\ (\to\Box^i)$$

where Σ_1 and Σ_2 are empty or consist of propositional variables only, i is some index or $i = \varnothing$, (m) is some mark or $(m) = \varnothing$.

What is more, transitivity rule can only be applied if either $(m) = \varnothing$ or $\Box\Gamma$ contains at least one formula of the form $[\Box H]^l$ where $m \leqslant l \leqslant n-1$.

Analogous changes are introduced to $G^{*}S4^c_n$ to obtain the terminating calculus for logic $K4^c_n$.

Definition 3.5.4. *A sequent is called labelled for $K4^c_n$, if every indexable occurrence of $\Box_l$ is indexed with integer and all the formulas are numbered.*

Definition 3.5.5. *Labelling for $K4^c_n$ of sequent S is denoted* $\mathrm{Lab}_{K4^c_n}(S)$ *and labelled for $K4^c_n$ sequent* $\mathrm{Lab}_{K4^c_n}(S)$ *is obtained from S by (1) indexing all the indexable occurrences of $\Box_l$ with different natural numbers, and (2) attaching number* 1 *to every formula. No marks are needed in* $\mathrm{Lab}_{K4^c_n}(S)$.

Now the calculus is defined as follows.

Definition 3.5.6. *The labelled sequent calculus without loops for logic* $K4_n^c$ *(*$G^{*}K4_n^c$*) consists of axiom* $\Gamma, [F_1]^n \to [F_2]^n, \Delta$, *where* $\mathrm{Proj}(F_1) = \mathrm{Proj}(F_2)$, *the same logical and simplification rules as in* $G^{*}S4_n^c$ *and rules:*

Transitivity:

$$\frac{\left\{[\Gamma_1^{i\leftarrow n}]^n, \Box_a\Gamma_1^{i\leftarrow n} \to [F]^n\right\}^{\neq a:\emptyset}}{\Box_a\Gamma_1, \Box_{\neq a}\Gamma_2, \Sigma_1 \to \Sigma_2, \Box_*\Delta, [\Box_a^{i(m)}F]^{n-1}} \; (\to \Box_a^i)$$

where Σ_1 *and* Σ_2 *are empty or consist of propositional variables only,* $a \neq c$, i *is some index or* $i = \varnothing$, (m) *is some mark or* $(m) = \varnothing$. *Sequent* $\{S\}^{\neq a:\emptyset}$ *is obtained from* S *by replacing all the occurrences of* $\Box_l^{j(m')}$ *to* $\Box_l^j$ *for every* $l \neq a$, *every* j *and every* m'.
What is more, transitivity rule can only be applied if either $(m) = \varnothing$ *or* $\Box_a\Gamma_1$ *contains at least one formula of the form* $[\Box_a C]^{n_1}$ *where* $m \leqslant n_1 \leqslant n-1$.
Central agent:

$$\frac{[\Gamma^{i\leftarrow n}]^n, \Box_*\Gamma^{i\leftarrow n} \to [F]^n}{\Box_*\Gamma, \Sigma_1 \to \Sigma_2, \Box_*\Delta, [\Box_c^{i(m)}F]^{n-1}} \; (\to \Box^{c,i})$$

where Σ_1 *and* Σ_2 *are empty or consist of propositional variables only,* i *is some index or* $i = \varnothing$, (m) *is some mark or* $(m) = \varnothing$.
Once again, central agent rule can only be applied if either $(m) = \varnothing$ *or* $\Box_*\Gamma$ *contains at least one formula of the form* $[\Box_l C]^{n_1}$ *for some* l, *where* $m \leqslant n_1 \leqslant n-1$.

The proofs of the soundness, completeness and derivation search termination of both $G^{*}K4$ and $G^{*}K4_n^c$ are analogous to the respective proofs for $G^{*}S4$ and $G^{*}S4_n^c$ and require only small and obvious changes.

Conclusions

In this dissertation new sequent calculi for multimodal logic K_n^c, T_n^c, $K4_n^c$ and $S4_n^c$ with central agent axiom are presented. These calculi are obtained from regular multimodal calculi by adding the rule for central agent axiom. The similarity of central agent rule to rule $(\to \Box_l)$ ensures, that cut-elimination theorem for the calculi can be proved without much difficulty. This idea can also be used to develop sequent calculi for other multimodal logics with central agent axiom. It also can be extended to other similar axioms. For example, axioms of the form $\Box_{l_1} F \supset \Box_{l_2} F$.

Next, the work presents a new way to obtain termination in derivation search. This method uses four kinds of labels: stars, indexes, marks and formula numbers. Although in the dissertation this technique is applied only to sequent calculi of monomodal logics $K4$ and $S4$ and to multimodal logics $K4_n^c$ and $S4_n^c$, it can be used in other cases too. First of all, by removing central agent rule from the calculi for logics $K4_n^c$ and $S4_n^c$, Gentzen-type calculi for regular multimodal logics $K4_n$ and $S4_n$ are obtained. Next, it is possible to apply this method for multimodal logics with different interaction axioms. Moreover, the separate ideas of this method can be adapted to other multimodal logics. For example, stars in this dissertation were used to ensure the termination of derivation search in sequent calculi for multimodal logics T_n^c.

Finally, as mentioned in the dissertation, the knowledge of central agent is actually a distributed knowledge of other agents. Therefore, all the techniques used in this thesis can be employed in developing terminating sequent calculi for logics with distributed knowledge.

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Appendix A

Proof of Lemma 2.3.6

In this appendix the formal proofs of derivability of formulas, presented in Lemma 2.3.6, are provided. The appendix is organized in the same manner as the formulation of the lemma. Therefore, it is divided into two sections according to the set of calculi in which the formulas are derivable. Moreover the order of proofs is the same as the order of the formulas in Lemma 2.3.6.

A.1 Formulas Derivable in $\boldsymbol{HK^c_n}$, $\boldsymbol{HK4^c_n}$, $\boldsymbol{HT^c_n}$ and $\boldsymbol{HS4^c_n}$

The formulas, discussed in this section, are derivable in all the defined Hilbert-type calculi with central agent axiom. Therefore, for the proofs of the derivability only the axioms and rules of HK^c_n can be used, because they are also part of other calculi.

Lemma A.1.1. *Formula* $(F \vee G) \supset (G \vee F)$.

Proof. The derivation is simple:

1.	$F \supset (G \vee F)$	Axiom *3.2*, $\{G/F, F/G\}$.
2.	$G \supset (G \vee F)$	Axiom *3.1*, $\{G/F, F/G\}$.
3.	$(F \vee G) \supset (G \vee F)$	$R\vee$ rule from 1 and 2.

□

Lemma A.1.2. *Formula* $(F \supset G) \supset (\neg F \vee G)$.

Proof. This formula is a well known tautology of classical logic and it is easy to show that $\vDash_{\mathcal{L}} (F \supset G) \supset (\neg F \vee G)$ according to Definition 1.3.3 and the definition of validity, where $\mathcal{L} \in \{K^c_n, K4^c_n, T^c_n, S4^c_n\}$.

Indeed the reasoning does not depend on the logic used. For suppose contrary, that there is some structure $\mathcal{S} = \langle \mathcal{W}, \mathcal{R}_c, \mathcal{R}_1, \ldots, \mathcal{R}_n, \Phi \rangle$ and world $w \in \mathcal{W}$ such that $\mathcal{S}, w \nvDash (F \supset G) \supset (\neg F \vee G)$. Then according to the definition $\mathcal{S}, w \vDash F \supset G$ and $\mathcal{S}, w \nvDash \neg F \vee G$. From the latter it follows that $\mathcal{S}, w \nvDash \neg F$, therefore $\mathcal{S}, w \vDash F$, and $\mathcal{S}, w \nvDash G$. And this contradicts the fact that $\mathcal{S}, w \vDash F \supset G$, therefore formula $(F \supset G) \supset (\neg F \vee G)$ is true in every world of every Kripke structure and according to the definition it is valid in K_n^c, $K4_n^c$, T_n^c and $S4_n^c$.

Now from the completeness of the calculi $\mathcal{C}$, it follows immediately, that $\vdash_{\mathcal{C}} (F \supset G) \supset (\neg F \vee G)$, where $\mathcal{C} \in \{HK_n^c, HK4_n^c, HT_n^c, HS4_n^c\}$. $\square$

Lemma A.1.3. *Formulas:*

1. $(F \vee G) \supset (\neg F \supset G)$.

2. $(\neg F \wedge \neg G) \supset \neg(F \vee G)$.

3. $\neg(F \wedge G) \supset (\neg F \vee \neg G)$.

Proof. As well as in Lemma A.1.2, these formulas are also well known tautologies of classical logic and therefore valid in all the modal logics. Their derivability directly follows from the completeness of the calculi. Therefore the proof of this lemma is analogous to the proof of Lemma A.1.2. $\square$

Lemma A.1.4. *Formula* $(\Box_l F_1 \wedge \ldots \wedge \Box_l F_n) \supset \Box_l(F_1 \wedge \ldots \wedge F_n)$, *where* $n \geqslant 1$.

Proof. If $n = 1$, then the formula is $\Box_l F_1 \supset \Box_l F_1$ and it is derivable according to Example 1.1.6. Otherwise, the proof is by induction on n.

The base case. If $n = 2$, then the formula is $(\Box_l F_1 \wedge \Box_l F_2) \supset \Box_l(F_1 \wedge F_2)$ and the derivation of the formula is as follows:

1.	$F_1 \supset (F_2 \supset F_1)$	Axiom *1.1*, $\{F_1/F, F_2/G\}$.
2.	$(F_2 \supset F_1) \supset \Big((F_2 \supset F_2) \supset \big(F_2 \supset (F_1 \wedge F_2)\big)\Big)$	Axiom *2.3*, $\{F_2/F, F_1/G, F_2/H\}$.
3.	$F_1 \supset \Big((F_2 \supset F_2) \supset \big(F_2 \supset (F_1 \wedge F_2)\big)\Big)$	*Tr* rule from 1 and 2.
4.	$(F_2 \supset F_2) \supset \big(F_1 \supset (F_2 \supset F_2)\big)$	Axiom *1.1*, $\{F_2 \supset F_2/F, F_1/G\}$.
5.	$F_2 \supset F_2$	As in Example 1.1.6.
6.	$F_1 \supset (F_2 \supset F_2)$	*MP* rule from 5 and 4.
7.	$F_1 \supset \big(F_2 \supset (F_1 \wedge F_2)\big)$	*R*$\supset$ rule from 3 and 6.
8.	$\Box_l\Big(F_1 \supset \big(F_2 \supset (F_1 \wedge F_2)\big)\Big)$	NG_l rule from 7.
9.	$\Box_l F_1 \supset \Box_l\big(F_2 \supset (F_1 \wedge F_2)\big)$	K_l rule from 8.
10.	$\Box_l\big(F_2 \supset (F_1 \wedge F_2)\big) \supset \big(\Box_l F_2 \supset \Box_l(F_1 \wedge F_2)\big)$	Axiom *(K_l)*, $\{F_2/F, (F_1 \wedge F_2)/G\}$.
11.	$\Box_l F_1 \supset \big(\Box_l F_2 \supset \Box_l(F_1 \wedge F_2)\big)$	*Tr* rule from 9 and 10.
12.	$(\Box_l F_1 \wedge \Box_l F_2) \supset \Box_l F_1$	Axiom *2.1*, $\{\Box_l F_1/F, \Box_l F_2/G\}$.
13.	$(\Box_l F_1 \wedge \Box_l F_2) \supset (\Box_l F_2 \supset \Box_l(F_1 \wedge F_2))$	*Tr* rule from 12 and 11.
14.	$(\Box_l F_1 \wedge \Box_l F_2) \supset \Box_l F_2$	Axiom *2.2*, $\{\Box_l F_1/F, \Box_l F_2/G\}$.
15.	$(\Box_l F_1 \wedge \Box_l F_2) \supset \Box_l(F_1 \wedge F_2)$	*R*$\supset$ rule from 13 and 14.

The induction step. Suppose that the formula is derivable, if $n < k$. Let's show that formula $(\Box_l F_1 \wedge \ldots \wedge \Box_l F_{k-1} \wedge \Box_l F_k) \supset \Box_l(F_1 \wedge \ldots \wedge F_{k-1} \wedge F_k)$ is also derivable. The derivation is as follows:

1.	$(\Box_l F_1 \wedge \ldots \wedge \Box_l F_{k-1} \wedge \Box_l F_k) \supset (\Box_l F_1 \wedge \ldots \wedge \Box_l F_{k-1})$	Axiom *2.1*, $\{^{\Box_l F_1 \wedge \ldots \wedge \Box_l F_{k-1}}/_F, ^{\Box_l F_k}/_G\}$.
2.	$(\Box_l F_1 \wedge \ldots \wedge \Box_l F_{k-1}) \supset \Box_l(F_1 \wedge \ldots \wedge F_{k-1})$	Induction hypothesis ($n = k-1$).
3.	$(\Box_l F_1 \wedge \ldots \wedge \Box_l F_{k-1} \wedge \Box_l F_k) \supset \Box_l(F_1 \wedge \ldots \wedge F_{k-1})$	*Tr* rule from 1 and 2.
4.	$(\Box_l F_1 \wedge \ldots \wedge \Box_l F_{k-1} \wedge \Box_l F_k) \supset \Box_l F_k$	Axiom *2.2*, $\{^{\Box_l F_1 \wedge \ldots \wedge \Box_l F_{k-1}}/_F, ^{\Box_l F_k}/_G\}$.
5.	$(\Box_l F_1 \wedge \ldots \wedge \Box_l F_{k-1} \wedge \Box_l F_k) \supset \big(\Box_l(F_1 \wedge \ldots \wedge F_{k-1}) \wedge \Box_l F_k\big)$	$R\wedge$ rule from 3 and 4.
6.	$\big(\Box_l(F_1 \wedge \ldots \wedge F_{k-1}) \wedge \Box_l F_k\big) \supset \Box_l(F_1 \wedge \ldots \wedge F_{k-1} \wedge F_k)$	Induction hypothesis ($n = 2$).
7.	$(\Box_l F_1 \wedge \ldots \wedge \Box_l F_{k-1} \wedge \Box_l F_k) \supset \Box_l(F_1 \wedge \ldots \wedge F_{k-1} \wedge F_k)$	*Tr* rule from 5 and 6.

$\Box$

Lemma A.1.5. *Formula* $(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_n} F_n) \supset (\Box_c F_1 \wedge \ldots \wedge \Box_c F_n)$, *where* $n \geqslant 1$.

Proof. The proof is by induction on n.

The base case. If $n = 1$, then the formula is $\Box_{l_1} F_1 \supset \Box_c F_1$. If $l_1 \neq c$, then it is an axiom *(C)*. If however $l_1 = c$, then it is derivable according to Example 1.1.6.

The induction step. Suppose that the formula is derivable, if $n < k$. Let's show that formula $(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_k} F_k) \supset (\Box_c F_1 \wedge \ldots \wedge \Box_c F_k)$ is also derivable. The derivation is as follows:

1.	$(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1} \wedge \Box_{l_k} F_k) \supset (\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1})$	Axiom *2.1*, $\{^{\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1}}/_F, ^{\Box_{l_k} F_k}/_G\}$.
2.	$(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1}) \supset (\Box_c F_1 \wedge \ldots \wedge \Box_c F_{k-1})$	Induction hypothesis ($n = k-1$).
3.	$(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1} \wedge \Box_{l_k} F_k) \supset (\Box_c F_1 \wedge \ldots \wedge \Box_c F_{k-1})$	*Tr* rule from 1 and 2.
4.	$(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1} \wedge \Box_{l_k} F_k) \supset \Box_{l_k} F_k$	Axiom *2.2*, $\{^{\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1}}/_F, ^{\Box_{l_k} F_k}/_G\}$.
5.	$\Box_{l_k} F_k \supset \Box_c F_k$	Induction hypothesis ($n = 1$).
6.	$(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1} \wedge \Box_{l_k} F_k) \supset \Box_c F_k$	*Tr* rule from 4 and 5.
7.	$(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1} \wedge \Box_{l_k} F_k) \supset (\Box_c F_1 \wedge \ldots \wedge \Box_c F_{k-1} \wedge \Box_c F_k)$	$R\wedge$ rule from 3 and 6.

$\Box$

A.2 Formula Derivable in $\boldsymbol{HK4_n^c}$ and $\boldsymbol{HS4_n^c}$

This formula is derivable only in transitive modal logics. Therefore, only axioms and rules of $HK4_n^c$ can be used in the proofs, because they are also part of $HS4_n^c$.

Lemma A.2.1. *Formula* $(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_n} F_n) \supset (\Box_{l_1}\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_n}\Box_{l_n} F_n)$, *where* $n \geq 1$.

Proof. By induction on n.

The base case. If $n = 1$, then the the formula is $\Box_{l_1} F_1 \supset \Box_{l_1}\Box_{l_1} F_1$ and it is an axiom *(4_{l_1})* of $HK4^c_n$ and $HS4^c_n$.

The induction step. Suppose that the formula is derivable, if $n < k$. Let's show that it is also derivable for $n = k$. The derivation is as follows:

1.	$(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1} \wedge \Box_{l_k} F_k) \supset (\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1})$	Axiom *2.1*, $\{^{\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1}}/_F, {}^{\Box_{l_k} F_k}/_G\}$.
2.	$(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1}) \supset (\Box_{l_1}\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}}\Box_{l_{k-1}} F_{k-1})$	Induction hypothesis ($n = k - 1$).
3.	$(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1} \wedge \Box_{l_k} F_k) \supset (\Box_{l_1}\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}}\Box_{l_{k-1}} F_{k-1})$	*Tr* rule from 1 and 2.
4.	$(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1} \wedge \Box_{l_k} F_k) \supset \Box_{l_k} F_k$	Axiom *2.2*, $\{^{\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1}}/_F, {}^{\Box_{l_k} F_k}/_G\}$.
5.	$\Box_{l_k} F_k \supset \Box_{l_k}\Box_{l_k} F_k$	Induction hypothesis ($n = 1$).
6.	$(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1} \wedge \Box_{l_k} F_k) \supset \Box_{l_k}\Box_{l_k} F_k$	*Tr* rule from 4 and 5.
7.	$(\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}} F_{k-1} \wedge \Box_{l_k} F_k) \supset (\Box_{l_1}\Box_{l_1} F_1 \wedge \ldots \wedge \Box_{l_{k-1}}\Box_{l_{k-1}} F_{k-1} \wedge \Box_{l_k}\Box_{l_k} F_k)$	*R*$\wedge$ rule from 3 and 6.

$\Box$