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https://orcid.org/0000-0002-5916-9463

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Aidas Medžiūnas

Various Classes of Stochastic Differential Equations: Existence, Uniqueness, and Approximation

DOCTORAL DISSERTATION

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Academic supervisor – Prof. Habil. Dr. Kęstutis Kubilius (Vilnius University, Natural Sciences, Mathematics, N 001).

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Members:
Prof. Habil. Dr. Vydas Čekanavičius (Vilnius University, Natural Sciences, Mathematics, N 001),
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Prof. Dr. Olga Štikonienė (Vilnius University, Natural Sciences, Mathematics, N 001),
Assoc. Prof. Dr. Kostiantyn Ralchenko (Taras Shevchenko National University of Kyiv, Natural Sciences, Mathematics, N 001).

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Address: Naugarduko st. 24, Vilnius, Lithuania.
Tel. +370 219 3050; e-mail: mif@mif.vu.lt.
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VILNIAUS UNIVERSITETAS

Aidas Medžiūnas

Įvairios stochastinių diferencialinių lygčių klasės: egzistavimas, vienatis ir aproksimacija

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Gynimo taryba:
Pirmininkas – prof. dr. Jonas Šiaulys (Vilniaus universitetas, gamtos mokslai, matematika, N 001).
Nariai:
prof. habil. dr. Vydas Čekanavičius (Vilniaus universitetas, gamtos mokslai, matematika, N 001),
doc. dr. Viktor Skorniakov (Vilniaus universitetas, gamtos mokslai, matematika, N 001),
prof. dr. Olga Štikonienė (Vilniaus universitetas, gamtos mokslai, matematika, N 001),
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tel. +370 219 3050; el. paštas mif@mif.vu.lt

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Dedicated to R.

Contents

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Notation and abbreviations

Notation

$\mathbb{N}$	the set of positive integers
$\mathbb{Z}$	the set of integers
$\mathbb{R}$	the set of reals
C	space of continuous $\mathbb{R}$-valued functions
C^λ	space of λ-Hölder continuous functions
Ω	sample space
$\mathcal{A}$	σ-algebra of Ω subsets
P	probability measure on $\mathcal{A}$
$\mathbb{E}$	expected value
$\Delta_{n,k}^{(2)}$	second order increments
$\mathcal{N}$	Gaussian distribution
W	Standard Brownian motion/Wiener process
B^H	Fractional Brownian motion (fBm) with $H \in (0,1)$
$\stackrel{D}{=}$	Equality in distribution

Abbreviations

BEM	backward Euler-Maruyama
CIR	Cox-Ingersoll-Ross
CIR VR	Cox–Ingersoll–Ross variable-rate
CKLS	Chan–Karolyi–Longstaff–Sander
fBm	fractional Brownian motion
fSDE	fractional stochastic differential equation
LBE	Lamperti-backward Euler
SDE	stochastic differential equation

Chapter 1

Introduction

Stochastic processes based modelling has a long and rich history in a variety of fields. During the last several decades, there was an effort to expand classical stochastic differential equations (SDEs) driven by the Wiener process (standard Brownian motion) into SDEs driven by fractional Brownian motion (fBm) (see, e.g. [47, 48, 58], etc.). Since fBm B^H is a generalization of a Wiener process W and introduces new valuable properties of short/long memory (see [24, 54]) into SDEs, it greatly expands the modelling possibilities of a variety of processes observed in finance, medicine, biology, physics (see, e.g. [5, 6, 51, 56]).

For instance, the application of fractional SDEs allows for a more accurate representation of asset price dynamics, capturing long-range dependence and volatility clustering observed in financial markets. This enhanced modelling framework has proven beneficial in statistical arbitrage, market efficiency investigations in econophysics, and portfolio optimization (e.g., [13, 16, 22]).

Similarly, in medicine and biology, fBm-driven SDEs have been utilised to model complex physiological processes, such as the movement of cells, the spread of diseases, or the behaviour of biomolecules. The inclusion of fractional Brownian motion provides a means to incorporate memory effects in various biological systems (e.g., [17, 28, 40]).

In this work, we will study various classes of diffusion stochastic differential equations (SDEs) in the form:

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dZ_s, \qquad t \in [0, T], \tag{1.1}$$

where $Z = (Z_t)_{t \geqslant 0}$, $Z_0 = 0$ is an arbitrary stochastic process, with continuous paths, $b, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ are some continuous functions.

We can interpret the stochastic integral in Eq. (1.1) as either a pathwise Riemann–Stieltjes integral or as Ito integral (dependent on the nature of Z). For example, as a special case, we can take $Z = B^H$, where B^H is a fBm.

The classification of diffusion stochastic differential equations is essential for understanding and studying their behaviour. It allows us to categorise different types of SDEs based on their key characteristics, such as the drift coefficient b, diffusion coefficient σ, random process Z in the stochastic integral, initial conditions, and any additional constraints on the solution X.

By examining the properties of these coefficients and processes, we can identify broad classes of SDEs that share similar mathematical structures and properties. For instance, a class of SDEs may have a linear drift coefficient and a constant diffusion coefficient, while another class might have a nonlinear drift and a time-varying diffusion coefficient.

Once the classes are defined, we can further investigate and analyse more specific sub-classes within each class. These sub-classes are often referred to as models, which capture more detailed characteristics and assumptions about the system under study. It should be noted that the distinction between class and model is not rigid, but rather a practical and conceptual tool.

The history and taxonomy of stochastic models are wide and deep, and in his thesis, we will be concerned only with a few of them. I.e. Chan–Karolyi–Longstaff–Sanders (CKLS) model, which is used to approximate prices of bonds and bond options, currency exchange rates or contingent claims (e.g., [8, 49, 53]); Ornstein-Ulenbeck and Vasicek processes which have a wide range of applications from analysis of the evolution of phenotypic traits to interest rates (e.g., [4, 21]); Pearson diffusion model, which is closely related to Fokker–Planck equation and is applied in physical and chemical sciences, engineering, rheology, environmental sciences, and financial mathematics (e.g., [39, 50]).

Finally, one of the latest new key aspects in investigations of SDEs, which opened intriguing new horizons both for this thesis and the SDE field in general, is the influence of existing boundaries (walls) on properties of fBms and SDEs [20]. The majority of SDEs classes and models discussed above propagate *free* (i.e. not limited by surrounding medium). Usually, it is a reasonable assumption in finance, but it is not so in natural sciences (e.g. if one wants to model the growth of serotonergic axons in the brainstem [27, 55, 56] or angiogenesis in a tumour [7]). Therefore a whole spectrum of these wall conditions can be introduced into SDEs based on their properties (e.g. elastic wall, inelastic wall, sticky wall [55]).

Furthermore, a natural bridge can be drawn between these SDEs with walls and research of SDEs with positive solutions [3, 10, 19, 46, 52, 57]. One simply has to interpret the positivity condition as a motion of a random process limited to the semi-infinite axis (i.e. $X \in (0, \infty)$ or $X \in [0, \infty)$) by reflecting wall at 0. Therefore, all the research presented in this thesis becomes nicely bound by the idea of boundaries for SDEs.

1.1 Aims and problems

We give a short summary of the problems considered in this work.

In Chapter 3 we study a class of fractional stochastic differential equations (fSDEs) with coefficients that may not satisfy the linear growth condition and non-Lipschitz diffusion coefficient. Our goal is to obtain conditions for the positivity of such fSDEs solutions and show the almost sure convergence rate of the backward Euler approximation scheme when applied to them. Moreover, since for any fSDE, the Hurst index is one of the fundamental parameters, we want to obtain a high-quality estimator of this parameter for positive solutions of fSDEs being investigated. Furthermore, for our results to be practically applicable, concrete examples of classical models satisfying the conditions of our theorems have to be found.

In Chapter 4 we study one-dimensional stochastic differential equations (SDEs) driven by a stochastic process with Hölder continuous paths of order $1/2 < \gamma < 1$. Since the solution of such SDEs does not have an explicit form, an approximation scheme for its solution has to be constructed and its convergence rate found. Naturally, for the approximation scheme to be valuable it has to have a higher convergence rate than the rates achieved by other authors in the previous research. Again, for the results to be useful in practical applications, one has to find examples of classical models fitting SDEs researched in Chapter 4. Furthermore, in Chapter 5 we explore an integrated fractional Brownian motion and search for its approximation scheme.

In Chapter 6 we introduce a fractional SDE with a soft wall. It has been established that SDEs with reflection can be imagined as equations having a hard wall. Now by introducing repulsion instead of reflection, one gets an SDE with a soft wall. In contrast to the SDE with reflection, where the process cannot pass the hard wall, the soft wall is repulsive but not impenetrable. As the process crosses the soft wall boundary, it experiences a force of a chosen magnitude in the opposite direction. When the process is far from the wall, the force acts weakly. We seek to form a stochastically rigid SDE-based model, that encompasses all these behaviours, find conditions under which it has a unique solution and construct an implicit Euler approximation with a high rate of convergence for this equation. Moreover, due to the novelty of the model, it is crucial to demonstrate the particularities of its behaviour using tools of mathematical modelling.

1.2 Methods

In this thesis, we employ a range of commonly and less commonly in the field of stochastic differential equations used techniques.

Since the interaction induced by the diffusion function between random noise and process in SDEs investigated in Chapters 3 and 4 is usually unwanted, we use the Lamperti transform, which introduces supplementary SDE with a very simple diffusion function without this interaction. Naturally, in order to use this transform certain conditions on the diffusion component had to be imposed, and compatibility between the solution of initial SDE, supplementary solution, and their approximations was shown.

All three classes of SDEs investigated in Chapters 3, 4 and 6 do not have solutions, which can be expressed in explicit form. Hence, various approximation schemes had to be constructed. These range from classical implicit and explicit Euler-type schemes to more original Milstein-like approximations.

When proving the existence of a solution for the SDE class with a soft wall in Chapter 6 we applied the implicit Picard iteration method. We found this method most applicable because it does not introduce new restrictions on a repulsive force in the SDE when compared to other methods (e.g. explicit Picard iteration method [32]).

Asymptotics of integrated fractional Brownian motion approximation scheme in Chapter 5 were found by combining probability and number theory results. In particular, certain results about harmonic numbers, which (as far as we know) are a novel approach for SDEs research.

Estimates of Hurst index H for trajectories of SDEs in Chapters 3 and 6 were made using schemes based on second-order increments, quadratic variations, and their limiting behaviour.

1.3 Actuality and novelty

Findings presented in the thesis are original and correspond to a total of four publications in reputable mathematical journals [35, 36, 37, 44]. These findings both expand and improve on recent works of other authors by investigating classes of fSDEs, which encompass a wider range of stochastic models and propose new approximation schemes that demonstrate higher convergence rates for the solutions of these fSDEs. Furthermore, in some of our research, we construct and explore novel fSDEs, which can be of great interest to mathematicians and researchers in other fields like biology, medicine, or physics.

Chapter 2

Literature review

Let $(\Omega, \mathcal{A}, P)$ be a probability space. Here, Ω is a *sample space* (collection of all possible outcomes, experiment results); $\mathcal{A}$ – σ-algebra of Ω subsets; and P – probability measure on $\mathcal{A}$.

Function $X : \Omega \to \mathbb{R}$ is called a random variable if

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{A}, \forall x \in \mathbb{R}$$

and families of such functions $\{X_t, t \in \mathbb{T}\}$ are defined as random processes ($\mathbb{T}$ is usually a subset of $\mathbb{R}$ or $\mathbb{N}$ and in the most of the cases is called time).

2.1 Fractional Brownian motion

According to Kubilius et al. [38], Tudor [54] and others the first one to define a random process with properties of fractional Brownian motion was A.N. Kolmogorov [31] in 1940. Interestingly, Kolmogorov did not name these processes fractional Brownian motion but Wiener Spirals (in those early days of stochastic analysis his approach was much more geometry-inspired). However, the first fundamental study dedicated to fBm, giving its proper name (i.e. fractional Brownian motion) was published in 1968 by Benoit Mandelbrot and John Ness [43].

Definition 1. *Fractional Brownian motion (fBm) with Hurst index* $H \in (0,1)$ *is a Gaussian process* $B^H = \{B_t^H, t \in \mathbb{R}^+\}$ *having the following properties*

(i) $B_0^H = 0$;

(ii) $\mathbb{E}B_t^H = 0, \quad t \in \mathbb{R}^+$;

(iii) $\mathbb{E}B_t^H B_s^H = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \in \mathbb{R}^+.$

Using these fundamental properties one can easily prove the following basic characteristic behaviours of fractional Brownian motion:

Proposition 1. *If B^H is fractional Brownian motion, then the following is true:*

1. *(Self-similarity)* $\forall a > 0, \quad B_{at}^H \stackrel{D}{=} a^H B_t^H$;
2. *(Stationarity of increments)* $\forall h > 0, \quad B_{t+h}^H - B_t^H \stackrel{D}{=} B_h^H$;
3. *(Regularity)* $\forall \epsilon > 0, \exists r.v. C_\epsilon : |B_t^H - B_s^H| \leq C_\epsilon |t-s|^{H-\epsilon}$.

The regularity Condition 3 of fractional Brownian motion can further be expanded by the following two theorems:

Theorem 1 (Hölder-continuity of B^H (see [38], p. 4))**.** *Almost all sample paths of an fBm B^H are locally Hölder of order strictly less than $H \in (0,1)$. I.e. for all $T > 0$, there exists a non-negative random variable $G_{\gamma,T}$ such that $\mathbb{E}(|G_{\gamma,T}|^p) < \infty$ for all $p \geqslant 1$, and*

$$|B_t^H - B_s^H| \leqslant G_{\gamma,T}|t-s|^\gamma \quad a.s. \tag{2.1}$$

for all $s,t \in [0,T]$, where $\gamma \in (0,H)$.

Theorem 2 (Sample modulus for B^H (see [18], p. 48))**.** *The function $\rho_H(u) = u^H\sqrt{|\ln u|}$ for $u \geqslant 0$ is a sample modulus for B^H, that is, for almost all $\omega \in \Omega$, there is $K_\omega < \infty$ such that*

$$|B_t^H(\omega) - B_s^H(\omega)| \leqslant K_\omega \rho_H(|t-s|), \quad \text{for } t,s \in [0,T]. \tag{2.2}$$

2.1.1 Wiener process vs. Fractional Brownian motion

Note that for the Hurst index value $H = \frac{1}{2}$ fractional Brownian motion Definition 1 turns into Wiener process (usually denoted W) definition (independence of the increments can be demonstrated pretty easily).

Definition 2. *We will consider real-valued stochastic process $W = \{W_t, t \geq 0\}$ to be Wiener process (standard Brownian motion) if it satisfies these conditions:*

(i) $W_0 = 0$;

(ii) *Process W has independent increments;*

(iii) *Increments $W_t - W_s$ have Gaussian distribution with mean 0 and variance $t-s$ (i.e. $W_t - W_s \sim \mathcal{N}(0, t-s)$).*

At first glance the connection between Wiener process W and fractional Brownian motion B^H seems to be very natural and tight. However, in the majority of the research, it could not be further from the truth. The independence of Wiener process increments on the one hand and dependence of fractional Brownian motion increments, on the other hand, has fundamental repercussions. Not only most of the theoretical and practical framework used in Wiener process research cannot be directly applied to fractional Brownian motion, but also this dependence fundamentally changes the nature of the process itself.

One of the best examples of this difference in process nature is the memory phenomenon demonstrated by fractional Brownian motions. There are many different ways to define concretely what the process memory is, but in general, one can say that the particular process has memory, if it is non-Markovian (i.e. not all information about process behaviour is collected in the present). Hence

Definition 3. *We say that process X exhibits long-range dependence or is a long-memory process if*

$$\sum_{i \geq 0} r_i = \infty$$

and exhibits short-range dependence or is a short-memory process if

$$\sum_{i \geq 0} r_i < \infty,$$

where $r_i = \mathbb{E}(X_1 - X_0)(X_{i+1} - X_i)$.

Since for fractional Brownian motion we have

$$r_i = \frac{1}{2}\left(\left((i+1)^{2H} - i^{2H}\right) - \left(i^{2H} - (i-1)^{2H}\right)\right), \quad i \geq 1, \quad r_0 = 1,$$

which after telescoping gives us $2\sum_{i=0}^{n} r_i = (n+1)^{2H} - n^{2H} + 1$, which for large enough n behaves as $(n^{2H})' = 2Hn^{2H-1}$. I.e. converges when $H \leq \frac{1}{2}$ and diverges when $H > \frac{1}{2}$. Thus, respectively, one gets that fractional Brownian motion has long-range memory for $H > \frac{1}{2}$ and short-range memory for $H \leq \frac{1}{2}$.

It has to be noted that one should not be misled by the Definition 3, although for Wiener process $\sum_{i\geq 0} r_i < \infty$, it is memoryless due to independence of increments.

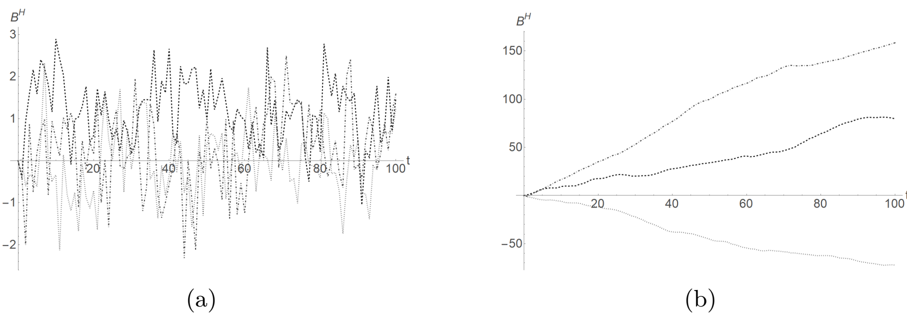
2.1.2 Practical considerations about Hurst index

While considering the effects of the Hurst index on trajectories of fractional Brownian motion one can identify two main ways to approach it.

The first one can be attributed more to the original works of British hydrologist Harold Edwin Hurst himself [24], i.e. the influence of H on the trend of B_t^H trajectories/series (especially seen well over *long* time periods):

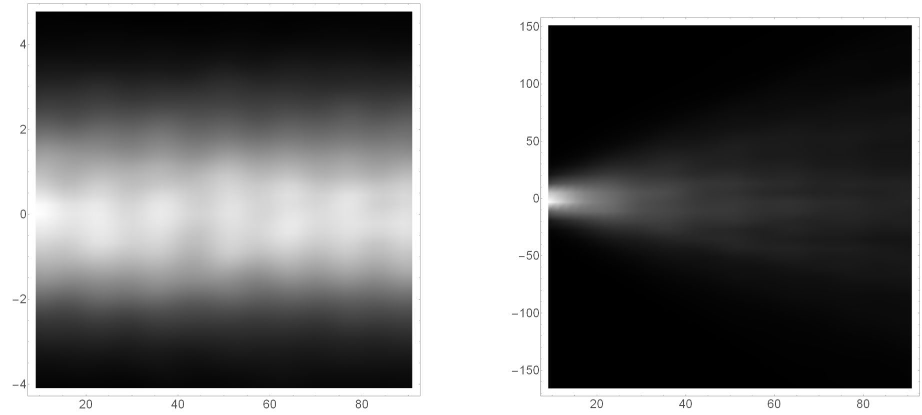
1. $H < \frac{1}{2}$ - mean-reverting (anti-persistent) behaviour. The positive/negative increment of process value will most likely be followed by the negative/positive increment of process value. Consequently, the process trajectories will oscillate around the mean $\mathbb{E}B^H$ (see Fig. 2.1(a) and Fig. 2.2(a)).

2. $H > \frac{1}{2}$ - trending (persistent) behaviour. The positive/negative increment of process value will most likely be followed by another positive/negative increment of process value. Consequently, the process trajectories will tend to wander away from the mean $\mathbb{E}B^H$ (see Fig. 2.1(b) and Fig. 2.2(b)).



(a) (b)

Figure 2.1: Sample trajectories of fractional Brownian motion.
(**a**) Hurst index $H = 0.05$. (**b**) Hurst index $H = 0.95$.



(a) (b)

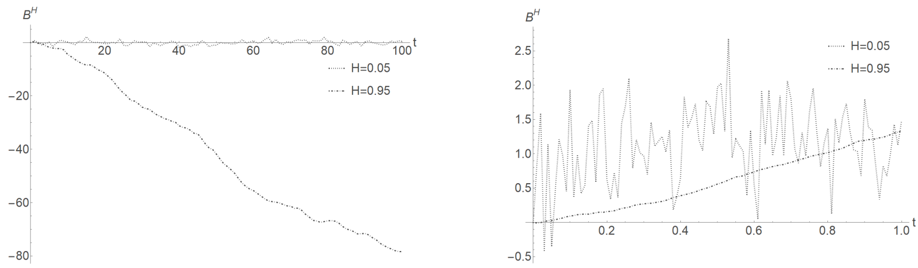
Figure 2.2: Heat maps of 1000 fractional Brownian motion trajectories. (**a**) Hurst index $H = 0.05$. (**b**) Hurst index $H = 0.95$.

The second way to look at Hurst index is more concerned with erraticism, irregularity, and nervousness of fractional Brownian motion trajectories and consequently the nature of its increments. Especially over *short* periods of time, since

$$B^H_{t^n_{k+1}} - B^H_{t^n_k} \overset{D}{=} \frac{T^H}{n^H} B^H_1, \tag{2.3}$$

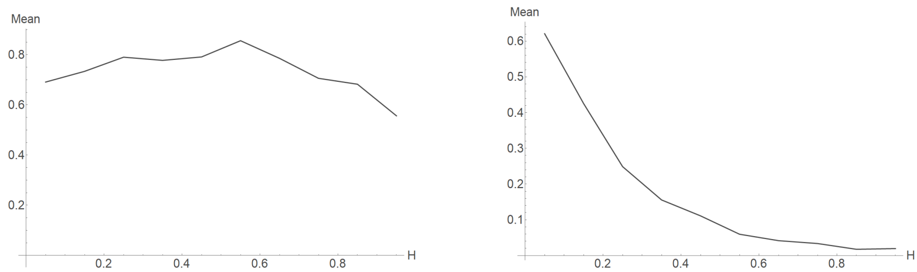
where $\pi = \{t^n_k = (k/n)T, 1 \leq k \leq n\}$ is a sequence of uniform partitions of the interval $[0, T]$.

Note how in Eq. (2.3) term n^{-H} in front of Brownian motion has an exponential dampening effect as H increases. Hence, the erraticism of fBm trajectories for *short* periods of time in Fig. 2.3(b) is much more contrasting than in Fig. 2.3(a) for *long* periods of time. We can prove this being no optical illusion by comparing the dependence of increments absolute value means with respect to the Hurst index (Fig. 2.4). As we can see, dependence is clearly exponential (as expected from Eq. (2.3)) for *short* periods of time (Fig. 2.4(b)) and negligible or even absent for *long* periods of time (Fig. 2.4(a)).



(a) (b)

Figure 2.3: Comparison of fBm trajectories. (**a**) *Long* periods of time. (**b**) *Short* periods of time.



(a) (b)

Figure 2.4: Means of absolute increments in respect to Hurst index. (**a**) *Long* periods of time. (**b**) *Short* periods of time.

2.1.3 Pathwise integration with respect to fractional Brownian motion

Naturally, another step to be taken while analyzing fractional Brownian motion is to inspect the issue of integration with respect to stochastic processes. In particular, due to the nature of fractional Brownian motion, there are a number of ways to do it, but for our research, the most important is one based on the Riemann–Stieltjes integral and p-variation calculus:

Definition 4. *Let $0 < p < \infty$. Then we will define the p-variation of function $f : [a.b] \to \mathbb{R}$ as*

$$v_p(f,[a,b]) := \sup_{\tau} \left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{t-1})|^p \colon t_i \in \tau \right\},$$

where $\tau = \{a = t_0^n < t_1^n < ... < t_{k_n}^n = b\}, n \in \mathbb{N}$, *are partitions of interval* $[a,b]$.

Note that p-variations can be infinite. Hence, we say that function f has finite p-variation if $v_p(f,[a,b]) < \infty$.

Definition 5. *Consider any sequence of partitions* $\tau = \{a = t_0^n < t_1^n < ... < t_{k_n}^n = b\}, n \in \mathbb{N}$ *such that* $\max_i |t_{i+1}^n - t_i^n| \to 0$, *and any sequence of intermediate subpartitions* $\xi^n = \{\xi_i^n \in [t_i^n, t_{i+1}^n], i = 0,1,...,k_n - 1\}, n \in \mathbb{N}$. *Then the Riemann-Stieltjes integral* $\int_a^b f(t)dg(t)$ *of a function* f *in an interval* $[a,b]$ *with respect to function* g *is the limit*

$$\int_a^b f(t)dg(t) := \lim_{n\to\infty} \sum_{i=0}^{k_n-1} f(\xi_i^n)(g(t_{i+1}^n) - g(t_i^n)), \tag{2.4}$$

provided this limit exists and does not depend on the choice of partitions τ *and* ξ.

In practice using this definition can be rather cumbersome or even impossible in some cases, thus, some kind of intermediate results are needed. For example, one classical proposition states that

Proposition 2. *[42] For the existence of Riemann-Stieltjes integral it suffices to show that* $f \in C[a,b]$ *and* g *has the finite* 1*-variation* ($i.e.$ $v_1(g,[a,b]) < \infty$) *of the interval* $[a,b]$.

However, fBm trajectories are nowhere differentiable and it can be easily shown [42] that this particular result is not applicable. Hence, another, more subtle result has to be used.

Denote $\mathcal{W}([a,b])$ the class of bounded p-variation functions over interval $[a,b]$. Then

Theorem 3 (Young-Stieltjes integrability theorem [59])**.**
Let $f \in \mathcal{W}_p([a,b])$ *and* $g \in \mathcal{W}_q([a,b])$ *with* $p,q > 0 : 1/p + 1/q > 1$. *If* f *and* g *have no common points of discontinuity then the Riemann-Stieltjes integral* $\int_a^b f dg$ *exists and, for all* $\xi \in [a,b]$ *the following inequality holds:*

$$\left| \int_a^b f dg - f(\xi)[g(b) - g(a)] \right| \leq \left(1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right) V_p(f;[a,b]) V_q(g;[a,b]),$$

where $\zeta(s) := \sum_{n\geq 1} n^{-s}$ *is the Riemann zeta function and*

$$V_p(\cdot;[a,b]) = v_p^{1/p}(\cdot;[a,b]).$$

Using the regularity property of fBm trajectories one can easily show that fractional Brownian motion has finite p-variation a.s., when $p > 1/H$. Therefore Theorem 3 is applicable for Riemann-Stieltjes integrals with respect to fractional Brownian motion.

Love–Young Inequality for Hölder continuous functions

Looking at Theorem 3 it should be no surprise that we can develop an approximation estimate for Riemann-Stieltjes integrals.

Definition 6. *$C([a,b]) : a < b$ denotes the space of continuous $\mathbb{R}$-valued functions defined on $[a,b]$. For $\lambda \in (0,1]$, $C^{\lambda}([a,b])$ denotes the space of Hölder continuous functions of order λ equipped with a norm*

$$\|f\|_{\lambda} := |f|_{\infty} + |f|_{\lambda}, \qquad |f|_{\lambda} = \sup_{\substack{s,t \in [a,b] \\ s \neq t}} \frac{|f(t) - f(s)|}{|s-t|^{\lambda}}, \qquad |f|_{\infty} = \sup_{t \in [a,b]} |f(t)|.$$

Remark 1. *Let $f \in C^{\lambda}([a,b])$ and $g \in C^{\mu}([a,b])$ with $\lambda + \mu > 1$. Then $f \in \mathcal{W}_{1/\lambda}([a,b])$, $g \in \mathcal{W}_{1/\mu}([a,b])$, $V_{1/\lambda}(f;[a,b]) \leq |f|_{\lambda}(b-a)^{\lambda}$, $V_{1/\mu}(g;[a,b]) \leq |g|_{\lambda}(b-a)^{\mu}$.*

Theorem 4 (Love-Young inequality (see [38], p. 10))**.** *Let $f \in C^{\lambda}([0,T])$ and $g \in C^{\mu}([0,T])$ with $\lambda + \mu > 1$. Then for any $y \in [0,T]$, Love–Young inequality has the form:*

$$\left| \int_0^T f\, dg - f(y)[g(T) - g(0)] \right| \leqslant \zeta(\lambda + \mu) |f|_{\lambda} \, |g|_{\mu} T^{\lambda + \mu}. \tag{2.5}$$

2.2 Stochastic differential equations

After this short introduction into the basic machinery of fractional Brownian motion and stochastic integration (in a pathwise sense at least), we can start investigating integral equations in the form:

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dZ_s, \qquad t \in [0,T], \tag{2.6}$$

where $Z = (Z_t)_{t \geqslant 0}$, $Z_0 = 0$, is an arbitrary stochastic process, with continuous paths, $b : [0,T] \times \mathbb{R} \to [0,\infty), \sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ are some continuous functions.

Very often (partially, for historical reasons) equation (2.6) can be found written down in the form

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dZ_t, \qquad X_0 = x_0. \tag{2.7}$$

It comes without saying that both forms in equation (2.6) and equation (2.7) are equivalent. Usually, function b is referred to as a drift coefficient and function σ as a diffusion coefficient.

2.2.1 Solution of SDE

Naturally, the great importance of stochastic analysis is put on search and analysis of SDE solution (process):

Definition 7. *A continuous random process $X_t, t \in [0, T]$, is called a solution of SDE (2.7) in an interval $[0, T]$, if $\forall t \in [0, T]$ equation (2.6) is satisfied with probability one.*

Solutions of equation (2.6) are often called diffusion processes. The existence of the SDE solution and its uniqueness mostly (but not exclusively) depends on the properties of functions b, σ and the type of *driving* stochastic process Z. Historically, most progress was made in exploring SDE solutions, where Z is Wiener process W. It should be noted that as a rule of thumb, existence and uniqueness conditions applicable to Wiener process SDEs do not work for fractional Brownian motion ones, and fractional SDEs existence and uniqueness condition sets are usually more restrictive and work for narrower classes of functions b, σ. Moreover, most fractional counterparts of classic SDEs simply do not have solutions in the explicit form.

2.2.2 Some stochastic models and their explicit solutions

Depending on the nature of functions b, σ, using SDE (2.7) we can describe a wide range of behaviors. In the following section, we will introduce stochastic models investigated in our research.

Vasicek and Ornstein-Uhlenbeck process

This family of models is one of the earliest classical stochastic models. Usually used to describe the fluctuations of interest rates, but it also can be utilized in many other fields such as biology, medical and environmental sciences [9].

We will define Vasicek process to be the unique solution to the following stochastic differential equation:

$$dX_t = (\theta_1 - \theta_2 X_t)\,dt + \theta_3\,dZ_t, \qquad X_0 = x_0, \tag{2.8}$$

with $\theta_3 \in \mathbb{R}^+$ and $\theta_1, \theta_2 \in \mathbb{R}$.

However, in finance, another parametrization of the same process is more prevalent:

$$dX_t = \theta(\mu - X_t)\,dt + \sigma\,dZ_t, \qquad X_0 = x_0, \tag{2.9}$$

where θ, μ, σ are interpreted as reversion (”decay-rate” or ”growth-rate”), long-run equilibrium value of the process and volatility, respectively (see Fig. 2.5).

For $Z = W$ Vasicek process has an explicit solution in the form of

$$X_t = \frac{\theta_1}{\theta_2} + \left(x_0 - \frac{\theta_1}{\theta_2}\right)\exp(-\theta_2 t) + \theta_3 \int_0^t \exp(-\theta_2(t-u))dW_u \tag{2.10}$$

or

$$X_t = \mu + (x_0 - \mu)\exp(-\theta t) + \sigma \int_0^t \exp(-\theta(t-u))dW_u, \tag{2.11}$$

depending on the parametrization.

Furthermore, the Vasicek process belongs to the class of SDEs, which actually have an explicit form of a solution for its fractional version. I.e. for $Z = B^H$ and $H \in (0,1)$ equation

$$X_t = x_0 + \int_0^t (\theta_1 - \theta_2 X_s)\,ds + \theta_3\,B_t^H, \quad t \geqslant 0,$$

with $\theta_3 \in \mathbb{R}_+$ and $\theta_1, \theta_2 \in \mathbb{R}$, has a unique solution (see [11], [41])

$$X_t = x_0 e^{-\theta_2 t} + \frac{\theta_1}{\theta_2}\left(1 - e^{-\theta_2 t}\right) + \theta_3 \int_0^t e^{\theta_2(s-t)} dB_s^H.$$

Ornstein–Uhlenbeck process is a somewhat simpler version of Vasicek process. It is a unique solution to the Langevin stochastic differential equation:

$$dX_t = \theta X_t\, dt + \sigma\, dZ_t, \quad X_0 = x_0, \tag{2.12}$$

where $\theta \in \mathbb{R}$, $\sigma > 0$.

Needless to say, for $Z = B^H$ and $H \in (0,1)$ Eq. (2.12) has a unique solution

$$X_t = x_0 e^{\theta t} + \sigma \int_0^t e^{\theta(t-s)} dB_s^H. \tag{2.13}$$

CKLS and CIR process

Chan–Karolyi–Longstaff–Sanders process can be considered a generalization of Vasicek model. This model is mostly used in finance (bond option pricing [29], predictions of interest rates [12]).

Define CKLS process to be the unique solution to the following stochastic differential equation:

$$dX_t = (\theta_1 - \theta_2 X_t)\, dt + \theta_3 X_t^\gamma\, dZ_t, \qquad X_0 = x_0 > 0, \tag{2.14}$$

with $\theta_1, \theta_3 \in \mathbb{R}^+$ and $\gamma \in [1/2, 1)$.

Same as Vasicek process, in finance, another parametrization is more frequent

$$dX_t = a(b - X_t)\, dt + \sigma X_t^\gamma\, dZ_t, \qquad X_0 = x_0 > 0, \tag{2.15}$$

where a is interpreted as mean reversion rate, b as mean of interest rate and σ as measure of volatility (see Fig. 2.6).

Another widely used model - Cox-Ingersoll-Ross model is a special case ($\gamma = 1/2$) of CKLS process:

$$dX_t = (\theta_1 - \theta_2 X_t)\, dt + \theta_3 \sqrt{X_t}\, dZ_t, \qquad X_0 = x_0 > 0, \tag{2.16}$$

with $\theta_1, \theta_2, \theta_3 \in \mathbb{R}^+$.

One interesting research aspect, which is introduced, when considering the research related to the CIR processes (other processes too, of course) is that the positivity of process trajectories is sometimes a very desirable or even vital feature. For instance, in mathematical finance CIR process is used for modelling interest rates, which naturally can be only positive. Hence, finding conditions, when solution positivity is ensured is very important. For instance, for $Z = W$, CIR process (2.16) is strictly positive when $2\theta_1 \geq \theta_3^2$ [46].

2.2.3 Approximation schemes of SDE solutions

It is important to stress that proving the existence and uniqueness of the solution does not ensure the possibility/ability to express this solution in explicit form. Very often it is simply impossible. Furthermore, as one can see, even in explicit solutions of the Vasicek process (see (2.10), (2.11)), the calculation of Riemann-Stieltjes integral is required, which (except trivial cases) introduces separate approximations of its own. Hence often some kind of approximation schemes have to be used. Specifically, in our research, we are concerned with discrete approximations of continuous SDE solution, which happen to be most popular and widely used in the field.

Family of Euler-Maruyama type approximations

One of the oldest discrete approximation schemes is a family of so-called Euler-Maruyama approximation methods. Let us begin by defining the explicit forward Euler-Maruyama method, first.

Definition 8. *The forward Euler-Maruyama approximation of SDE (2.7) solution X is a continuous stochastic process Y satisfying the following iterative scheme*

$$Y_{t_{i+1}} = Y_{t_i} + b(t_i, Y_{t_i})(t_{i+1} - t_i) + \sigma(t_i, Y_{t_i})(Z_{t_{i+1}} - Z_{t_i}),$$

where $t_i \in [0,T] : i = 0,...,n-1$.

Most often we use homogeneous time division ($\Delta t_i = t_{i+1} - t_i = T/n$).

Now, let $Z = W$ in Eq. (2.8). Then discrete time Euler-Maruyama approximation for homogeneous time subdivision

$$Y_{T(i+1)/n} = Y_{Ti/n} + \frac{(\theta_1 - \theta_2 Y_{Ti/n})T}{n} + \theta_3(W_{T(i+1)/n} - W_{Ti/n}) \quad (2.17)$$

has a strong convergence rate of $n^{-1/2}$ (see [25]).

Henceforth, as an example, using scheme (2.17) we can model Vasicek process trajectories:

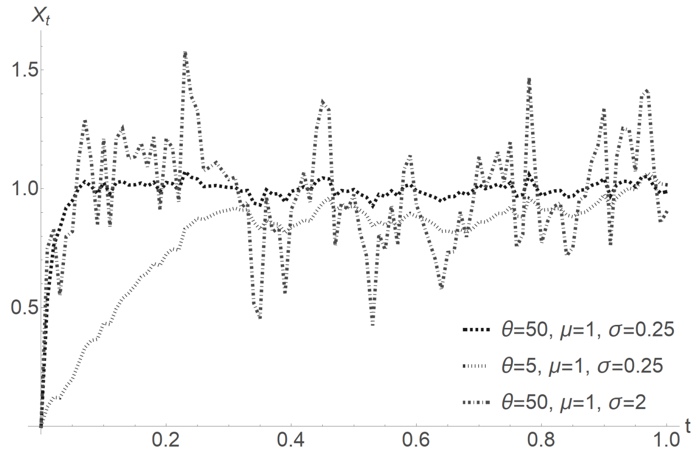


Figure 2.5: Wiener process driven Vasicek process trajectories modelled using Euler forward approximation scheme. Note trajectories behaviour change dependant on the values of θ, μ, σ.

However, although explicit, but forward Euler-Maruyama method is not always desirable due to stability issues, which arise for certain time divisions. Therefore, regularly the implicit backward Euler-Maruyama method is used.

Definition 9. *The backward Euler-Maruyama approximation of SDE (2.7) solution X is a continuous stochastic process Y satisfying the following iterative scheme*

$$Y_{t_{i+1}} = Y_{t_i} + b(t_{i+1}, Y_{t_{i+1}})(t_{i+1} - t_i) + \sigma(t_i, Y_{t_i})(Z_{t_{i+1}} - Z_{t_i}), \qquad (2.18)$$

where $t_i \in [0, T] : i = 0, ..., n-1$.

The Lamperti transform

Up a closer inspection of the CKLS family of models Eq. (2.14) one sees that (in contrast to Vasicek family Eq. (2.8)) the diffusion function σ is not a trivial constant anymore. It actually enforces an interaction between the state of the process and the random noise component [14]. This interaction is rarely wanted because it causes great difficulties in the stability of the solution and the well-definedness of approximation schemes. For example, direct application of Euler approximation schemes for CIR type of models (2.16) do not preserve the positivity. Hence, naturally, they are not very useful.

Some approaches, trying to avoid problems caused by these interactions (of non-trivial diffusion coefficient), propose transforming it directly using some other functions [15], putting limits on time divisions, or even redefining the whole approximation scheme [2]. However, all these approaches are rather heavy-handed, greatly limit the convergence rate of schemes themselves, and are not universal in general.

Therefore, one of the most efficient ways to avoid non-trivial diffusion coefficients is to get rid of them altogether. For instance, this can be achieved by using Lamperti transform:

Theorem 5 (Lamperti transform)**.** *Let $Z = W$ and X_t be an Ito-process defined as in Eq.2.7 with continuously differentiable and strictly positive/negative diffusion coefficient σ. Define Lamperti transform as function:*

$$\phi(X_t,t) = \int \frac{1}{\sigma(t,x)}dx\bigg|_{X_t}, \quad \phi(X_0,t) = 0. \tag{2.19}$$

If ϕ is injective function from process X_t state-space to $\mathbb{R}$ for $t \in [0,\infty)$, then $Y_t := \phi(X_t,t)$ is an injective mapping from process X_t state-space to $\mathbb{R}$ for $t \in [0,\infty)$ defined by the following stochastic differential equation

$$\begin{aligned} dY_t = &\left(\frac{\partial\phi(\phi^{-1}(t,Y_t),t)}{\partial t} + \frac{b(\phi^{-1}(t,Y_t),t))}{\sigma(\phi^{-1}(t,Y_t),t))} - \frac{1}{2}\frac{\partial\sigma(\phi^{-1}(t,Y_t),t)}{\partial X}\right)dt \\ &+ dW_t. \end{aligned} \tag{2.20}$$

As one can see Lamperti transform reduces any initial diffusion down to unit diffusion. However, it has to be stressed that for Lamperti transform to work properly it has to cover the whole range of the original diffusion function.

Moreover, for SDEs investigated in this thesis, we will need the version of Theorem 5, where $Z = B^H$. To achieve this, we present the following results

Theorem 6 (Chain rule (see [38], p. 10))**.**
Let $f = (f_1,\ldots,f_d) : [a,b] \to \mathbb{R}^d$ be a function such that for each $k = 1,\ldots,d$, $f_k \in C^\lambda([a,b])$, $\lambda \in (1/2,1]$. Let $g : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function with locally Lipschitz partial derivatives g'_k, $k = 1,\ldots,d$.

Then each $g_l' \circ f$ is Riemann–Stieltjes integrable with respect to f_k and

$$(g \circ f)(b) - (g \circ f)(a) = \sum_{k=1}^{d} \int_a^b (g_k' \circ f)\, df_k.$$

Proposition 3 (Substitution rule (see [38], p. 11))**.** *Let f, g, and h be functions in $C^{\lambda}([a,b])$, $\lambda \in (1/2, 1]$. Then for the Riemann–Stieltjes integral the following equality holds:*

$$\int_a^b f(x)\, d\Big(\int_a^x g(y)\, dh(y)\Big) = \int_a^b f(x) g(x)\, dh(x).$$

These results enable us to apply Lamperti transform for various SDEs with fractional Brownian motion.

Example 1. *Let*

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dB_s^H, \qquad t \in [0,T], \tag{2.21}$$

with strictly positive/negative diffusion coefficient σ.

Define a one-dimensional Lamperti transform as function:

$$Y_t = F(X_t) = \int_0^{X_t} \frac{1}{\sigma(x)} dx, \quad Y_0 = F(X_0). \tag{2.22}$$

Then Y_t satisfies the following SDE

$$Y_t = Y_0 + \int_0^t \frac{b(F^{-1}(Y_s))}{\sigma(F^{-1}(Y_s))} ds + B_t^H.$$

Lamperti-backward Euler approximation scheme

The establishment of backward Euler-Maruyama (BEM) approximation schemes and Lamperti transforms in combination gives birth to the so-called Lamperti-backward Euler (LBE) approximation scheme. One of the key insights/properties of the LBE approximation is that when applied, for example, to the CIR family of processes it ensures the positivity of the solution in a very natural way (compared to the methods applied in [2], [15]).

Let Lamperti transform $F(x) = \sqrt{x}$. Then after this transform CIR process (2.16) changes to

$$dY_t = \frac{1}{2}\theta_2 \left(\left(\frac{\theta_1}{\theta_2} - \frac{\theta_3^2}{4\theta_2} \right) Y_t^{-1} - Y_t \right) dt + \frac{1}{2}\theta_3 dW_t. \quad t > 0, \quad Y_0 = \sqrt{X_0}. \tag{2.23}$$

The BEM (see (2.18)) approximated unique solution of 2.23 is given by

$$Y_{t_{i+1}} = H^{-1}\left(Y_{t_i} + \frac{1}{2}\theta_3 (W_{t_{i+1}} - W_{t_i}) \right), \tag{2.24}$$

where $H(x) = x - \frac{1}{2}\theta_2 \left(\left(\frac{\theta_1}{\theta_2} - \frac{\theta_3^2}{4\theta_2} \right) x^{-1} - x \right) (t_{i+1} - t_i)$.

Define scheme p-strongly convergent with order one if

$$\mathbb{E} \sup_k |y(t_k) - Y_k|^p \leq C_p (\Delta t)^p,$$

where Y are values produced by the numerical method approximating some random process y and Δt is the length of interval for the uniform time division of this numerical approximation. Then Neuenkirch and Szpruch [46] proven that

Proposition 4. *Let $T > 0$ and $2 \leq p \leq \frac{4\theta_1}{3\theta_3^2}$. Then LBE scheme (2.24) of the CIR process (2.16) is p-strongly convergent with order one.*

Finally, the *original* CIR process trajectories are produced by reverting the Lamperti transform

$$X_{t_i} = F^{-1}(Y_{t_i}). \tag{2.25}$$

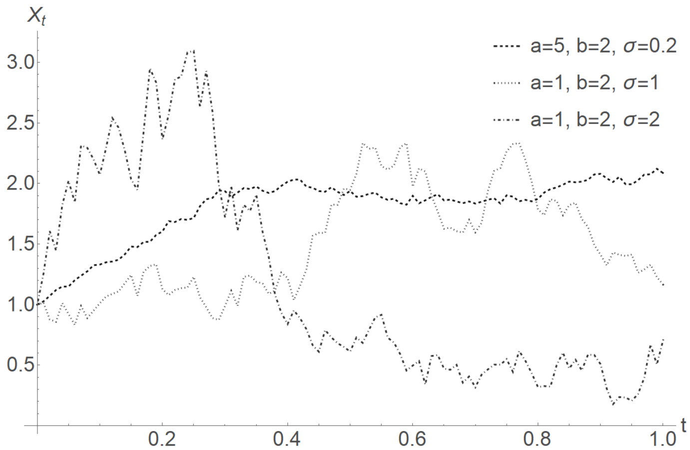


Figure 2.6: Wiener process driven CIR process trajectories modelled using Lamperti-Euler backward approximation scheme. Note trajectories behaviour changes dependant on the values of a, b, σ.

Chapter 3

Positive solutions of the fractional SDEs with non-Lipschitz diffusion coefficient

As discussed previously, positivity is an essential property in various financial models, including option pricing, stochastic volatility, and interest rate models. Therefore, establishing conditions that ensure the solutions remain positive is of great importance. It is crucial to preserve the positivity of approximate solutions when the true solution is positive because the non-negativity of numerical approximations is necessary for the scheme to be well-defined. For a unique global solution to an SDE to exist (i.e., without an explosion in finite time), the equation's coefficients generally need to meet linear growth and local Lipschitz conditions. However, these conditions are not satisfied in CKLS, CIR, Ait-Sahalia, Cox–Ingersoll–Ross variable-rate (CIR VR) models, to name a few.

In this chapter, we examine the SDE

$$Y_t = y_0 + (\beta - 1)\int_0^t h(Y_s)\,ds - (\beta - 1)\sigma B_t^H, \qquad \beta > 1, \tag{3.1}$$

where y_0 is a constant.

This equation is derived from the Lamperti transformation (see Section 2.2.3) of the FSDE

$$X_t = x_0 + \int_0^t g(X_s)\,ds + \sigma \int_0^t X_s^\beta\,dB_s^H, \qquad \beta \geqslant 1/2, \quad \beta \neq 1, \tag{3.2}$$

with $H \in (1/2, 1)$. The stochastic integral in equation (3.2) is a pathwise Riemann–Stieltjes integral (see Definition 5). Note that, SDE (3.2) cannot be treated directly since the functions $k(x) = x^\beta$, $\beta \geqslant 1/2$, $\beta \neq 1$, do not satisfy the usual Lipschitz conditions.

Our aim is to identify sufficiently simple conditions under which the solution of (3.1) is positive for $\beta > 1$ and $H \in (1/2, 1)$. By utilizing the backward Euler scheme (see Definition 9), which maintains positivity for (3.1), we achieve an almost sure convergence rate for X.

3.1 Main results

Our focus is on identifying the conditions under which the following SDE

$$X_t = x_0 + \int_0^t X_s^\beta f(X_s^{1-\beta})\, ds + \sigma \int_0^t X_s^\beta\, dB_s^H, \qquad \beta > 1,\ H \in (1/2, 1), \tag{3.3}$$

possesses a unique positive solution. The stochastic integral in equation (3.3) is a pathwise Riemann–Stieltjes integral.

3.1.1 Conditions

To state our main results, we assume that the following conditions on the function f in (3.3) are satisfied:

($\mathbf{C}_1$) f is locally Lipschitz on $(0, +\infty)$;

($\mathbf{C}_2$) There exist constants $a > 0$ and $\alpha \geqslant 0$ such that

$$\widehat{f}(x) := -f(x) \geqslant \frac{a}{x^{1+\alpha}}$$

for all sufficiently small $x \in (0, \infty)$;

($\mathbf{C}_3$) The function $\widehat{f}(x)$ satisfies *the one-sided Lipschitz condition,* that is, there exists a constant $K \in \mathbb{R}$ such that

$$(x - y)(\widehat{f}(x) - \widehat{f}(y)) \leqslant K(x - y)^2$$

for all $x, y \in (0, +\infty)$.

3.1.2 Theorems

Theorem 7. *If a function f satisfies conditions ($\boldsymbol{C}_1$)–($\boldsymbol{C}_3$), then equation* (3.3) *is well defined and has a unique positive solution $X \in \mathcal{C}^\gamma([0, T])$ of order $\gamma \in (\frac{1}{2}, H)$ for $H \in (\frac{1}{2}, 1)$, where $\mathcal{C}^\gamma([0, T])$ denotes the space of Hölder-continuous functions of order $\gamma > 0$ on $[0, T]$.*

Consider the SDE

$$Y_t = y_0 + (\beta - 1)\int_0^t \widehat{f}(Y_s)\,ds - (\beta - 1)\sigma B_t^H, \tag{3.4}$$

where $y_0 = x_0^{1-\beta}$, $t \geqslant 0$, $H \in (\frac{1}{2}, 1)$.

A strong approximation of the SDE that has locally Lipschitz drift for $H \in (\frac{1}{2}, 1)$ is constructed by applying the backward Euler scheme. By using the backward Euler scheme, which preserves positivity for (3.4), we obtain an almost sure convergence rate for X.

A sequence of uniform partitions of the interval $[0, T]$ we denote by $\pi^n = \{t_k^n = \frac{k}{n}T, 1 \leqslant k \leqslant n\}$ and let and $h = t_k^n - t_{k-1}^n$, $1 \leqslant k \leqslant n$. We introduce the backward Euler approximation scheme for Y

$$Y_{n,k+1} = Y_{n,k} + (\beta - 1)\widehat{f}(Y_{n,k+1})h - \sigma(\beta - 1)(B_{t_{k+1}^n}^H - B_{t_k^n}^H), \tag{3.5}$$

where $Y_{n,0} = y_0$, $0 \leqslant k \leqslant n - 1$.

The following assumption is needed for the positivity of the backward Euler approximation scheme to be preserved:

($\mathbf{C}_4$) Let $\widehat{F}(x) = x - (\beta - 1)\widehat{f}(x)h$ on $(0, \infty)$, where the function $\widehat{f}(x)$ satisfies condition ($\mathbf{C}_3$). There exists $h_0 > 0$ such that
$\lim_{x \to +\infty} \widehat{F}(x) = +\infty$ and $\lim_{x \to 0^+} \widehat{F}(x) = -\infty$ for $0 < h < h_0$.

Remark 2. *Note that under condition ($\boldsymbol{C}_3$), the function $\widehat{F}(x)$ is strictly monotone on $(0, \infty)$ for small h. This follows from ($\boldsymbol{C}_3$) and the inequality*

$$\begin{aligned}(x - y)(\widehat{F}(x) - \widehat{F}(y)) &= (x - y)^2 - (\beta - 1)(x - y)(\widehat{f}(x) - \widehat{f}(y))h \\ &\geqslant (1 - K^+(\beta - 1)h)(x - y)^2 > 0,\end{aligned}$$

where $K^+ = \max\{0, K\}$. Thus from the condition ($\boldsymbol{C}_4$) it follows that for each $b \in \mathbb{R}$, the equation $F(x) = b$ has a unique positive solution for $0 < h < h_0$. As a result, we see that the positivity is preserved by the backward Euler approximation scheme.

Definition 10. *Let (Z_n) be a sequence of r.v.s, let ς be an a.s. nonnegative r.v., and let $(a_n) \subset (0, \infty)$ be a vanishing sequence.*
Then $Z_n = O_\omega(a_n)$ means that $|Z_n| \leq \varsigma \cdot a_n$ for all n. Moreover,
$Z_n = O_\omega(1)$ means that the sequence (Z_n) is a.s. bounded.

Theorem 8. *Suppose that the function f in (3.3) is continuously differentiable on* $(0,+\infty)$ *and satisfies condition* $(\boldsymbol{C}_2)$ *and that there exists a constant $K \in \mathbb{R}$ such that the derivative is bounded above by K, that is, $f'(x) \leqslant K$. If the sequence of uniform partitions π of the interval* $[0,T]$ *is such that $h < h_0$, then for all $T > 0$ and $H \in (\frac{1}{2},1)$,*

$$\sup_{0\leqslant t\leqslant T} |Y_t - Y_t^n| = O_\omega(n^{-H}\sqrt{\ln n}), \tag{3.6}$$

where

$$Y_t^n = Y_{n,k} + \frac{t-t_k^n}{h}(Y_{n,k+1} - Y_{n,k})$$

and $t \in (t_k^n, t_{k+1}^n]$, $k = 0,\ldots,n-1$, $Y_{n,0}^n = y_0$. Moreover,

$$\sup_{0\leqslant t\leqslant T} |X_t - (Y_t^n)^{-1/(\beta-1)}| = \begin{cases} O_\omega(n^{-H}\sqrt{\ln n}) & \text{for } \beta \in (1,2], \\ O_\omega((n^{-H}\sqrt{\ln n})^{1/(\beta-1)}) & \text{for } \beta > 2, \end{cases} \tag{3.7}$$

where X is the solution of equation (3.3).

3.2 Proofs

The key approach for proving Theorem 7 hinges on the following proposition

Consider SDE

$$Y_t = y_0 + k_1 \int_0^t \widehat{f}(Y_s)\,ds + k_2 B_t^H, \qquad H \in (\tfrac{1}{2},1). \tag{3.8}$$

Proposition 5. *Suppose that a function f satisfies conditions* $(\boldsymbol{C}_1)$–$(\boldsymbol{C}_3)$. *If $y_0 > 0$, $k_1 > 0$, and $k_2 \in \mathbb{R}$, then there exists a unique positive solution of equation* (3.8) *such that $Y \in C^\gamma([0,T])$, $T > 0$, where $\gamma \in (\frac{1}{2}, H)$ and $H \in (\frac{1}{2},1)$.*

We easily see that the same proof as in Proposition 1 [34] remains valid for Proposition 5.

From Proposition 5 it follows that equation (3.4) has a unique positive solution.

Now, employing Proposition 5 as the principal instrument, we can prove the previously introduced theorems.

3.2.1 Proof of Theorem 7

Set $X_t = Y_t^{-1/(\beta-1)}$ and $x_0 = y_0^{-1/(\beta-1)}$, where Y is a solution of equation (3.4). Since the process Y is positive and continuous, for $s,t \in [0,T]$, we get

$$\begin{aligned}|Y_t^{-\beta/(\beta-1)} - Y_s^{-\beta/(\beta-1)}| &= \frac{|Y_t^{\beta/(\beta-1)} - Y_s^{\beta/(\beta-1)}|}{Y_t^{\beta/(\beta-1)} Y_s^{\beta/(\beta-1)}} \\ &\leqslant \frac{1}{\inf_{0\leqslant t\leqslant T} Y_t^{2\beta/(\beta-1)}} \frac{\beta}{\beta-1} \sup_{0\leqslant t\leqslant T} Y_t^{1/(\beta-1)} \cdot |Y_t - Y_s|.\end{aligned} \tag{3.9}$$

Thus X_s^β is a Hölder-continuous process up to the order $\gamma \in (\frac{1}{2}, H)$ on $[0,T]$. The process $X_t = Y_t^{-1/(\beta-1)}$ is the solution of equation (3.3). Indeed, by chain rule we obtain

$$\begin{aligned}X_t =& Y_t^{-1/(\beta-1)} = Y_0^{-1/(\beta-1)} - \frac{1}{\beta-1} \int_0^t Y_s^{-\beta/(\beta-1)}\, dY_s \\ =& y_0^{-1/(\beta-1)} - \frac{\beta-1}{\beta-1} \int_0^t Y_s^{-\beta/(\beta-1)} \widehat{f}(Y_s)\, ds \\ &+ \frac{\sigma(\beta-1)}{\beta-1} \int_0^t Y_s^{-\beta/(\beta-1)}\, dB_s^H \\ =& x_0 + \int_0^t g(X_s)\, ds + \sigma \int_0^t X_s^\beta\, dB_s^H.\end{aligned}$$

3.2.2 Proof of Theorem 8

We repeat the outlines of the proof of Theorem 3 in [34]. Note that under the conditions of the theorem, conditions $(\mathbf{C}_1)$–$(\mathbf{C}_3)$ are satisfied. Thus there exists a unique positive solution of SDE (3.4).

By the definition of Y^n, for any $t \in (t_k^n, t_{k+1}^n]$, we have

$$\begin{aligned}Y_t - Y_t^n =& Y_t - \frac{t - t_k^n}{h} Y_{n,k+1} - \frac{t_{k+1}^n - t}{h} Y_{n,k} \\ =& \frac{t_k^n - t}{h} (\beta-1) \Big[\int_t^{t_{k+1}^n} \widehat{f}(Y_s)\, ds - \sigma(B_{t_{k+1}^n}^H - B_t^H) \Big] \\ &+ \frac{t_{k+1}^n - t}{h} (\beta-1) \Big[\int_{t_k^n}^t \widehat{f}(Y_s)\, ds - \sigma(B_t^H - B_{t_k^n}^H) \Big] \\ &+ \frac{t - t_k^n}{h} (Y_{t_{k+1}^n} - Y_{n,k+1}) + \frac{t_{k+1}^n - t}{h} (Y_{t_k^n} - Y_{n,k}).\end{aligned}$$

Since the process Y is positive and continuous, from (2.2) it follows that

$$\begin{aligned}|Y_t-Y_s| \leqslant&(\beta-1)\int_s^t |\widehat{f}(Y_u)|\,du+(\beta-1)\sigma|B_t^H-B_s^H|\\ \leqslant&(\beta-1)(t-s)\sup_{0\leqslant u\leqslant T}|\widehat{f}(Y_u)|+(\beta-1)K_\omega|t-s|^H\sqrt{|\ln|t-s||}\\ =&O_\omega\Big(|t-s|^H\sqrt{|\ln|t-s||}\Big),\end{aligned}$$

and the asymptotic behavior of the first two terms is $O_\omega(n^{-H}\sqrt{\ln n})$. Thus it remains to obtain the asymptotics of the last two terms.

Note that

$$\begin{aligned}Y_{t_{k+1}^n}-Y_{n,k+1}=&Y_{t_k^n}-Y_{n,k}+(\beta-1)\int_{t_k^n}^{t_{k+1}^n}[\widehat{f}(Y_s)-\widehat{f}(Y_{n,k+1})]ds\\ &+(\beta-1)\zeta_{n,k+1}(Y_{t_{k+1}^n}-Y_{n,k+1})h,\end{aligned}$$

where $\zeta_{n,k+1}=\widehat{f}'(Y_{t_{k+1}^n}+\theta_{n,k+1}(\widehat{Y}_{n,k+1}-Y_{t_{k+1}^n}))$, $\theta_{n,k+1}\in(0,1)$. Then

$$\begin{aligned}&(Y_{t_{k+1}^n}-Y_{n,k+1})[1-(\beta-1)\zeta_{n,k+1}h]\\ &=Y_{t_k^n}-Y_{n,k}+(\beta-1)\int_{t_k^n}^{t_{k+1}^n}[\widehat{f}(Y_s)-\widehat{f}(Y_{n,k+1})]ds.\end{aligned}$$

Note that $1-\zeta_{n,k+1}(\beta-1)h\geqslant 1-(\beta-1)K^+h>0$
for $h<((\beta-1)K^+)^{-1}$ since $f'(x)\leqslant K$, where $K^+=\max\{0,K\}$. Thus

$$\begin{aligned}&|Y_{t_{k+1}^n}-Y_{n,k+1}|\\ &\leqslant\frac{1}{1-(\beta-1)K^+h}\left[|Y_{t_k^n}-Y_{n,k}|+(\beta-1)\int_{t_k^n}^{t_{k+1}^n}|\widehat{f}(Y_s)-\widehat{f}(Y_{t_{k+1}^n})|\,ds\right]\\ &=\sum_{i=1}^{k+1}I_i\prod_{j=i}^{k+1}(1-(\beta-1)K^+h)^{-1},\end{aligned}$$

where

$$I_i=(\beta-1)\int_{t_{i-1}^n}^{t_i^n}|\widehat{f}(Y_s)-\widehat{f}(Y_{t_i^n})|\,ds.$$

Applying the inequality $\ln\frac{1}{1-x}\leqslant\frac{x}{1-x}$, $x<1$, we get

$$\begin{aligned}\prod_{j=i}^{k+1}(1-(\beta-1)K^+h)^{-1}\leqslant&(1-(\beta-1)K^+h)^{-(k+2-i)}\\ \leqslant& e^{n\ln\frac{1}{1-(\beta-1)K^+h}}\leqslant e^{n\frac{(\beta-1)K^+h}{1-(\beta-1)K^+h}}\\ \leqslant& e^{\frac{(\beta-1)K^+T}{1-(\beta-1)K^+h}}.\end{aligned}$$

Further,

$$\begin{aligned}\int_{t_{k-1}^n}^{t_k^n}|\widehat{f}(Y_s)-\widehat{f}(Y_{t_k^n})|\,ds&\leqslant Tn^{-1}\max_{0\leqslant t\leqslant T}|\widehat{f}'(Y_t)|\max_{t_{k-1}^n\leqslant t\leqslant t_k^n}|Y_t-Y_{t_k^n}|\\ &=O_\omega(n^H\sqrt{\ln n}).\end{aligned}$$

This finishes the proof of (3.6).

It remains to prove (3.7). We will use the well-known inequalities

$$\begin{aligned}&|x^p-y^p|\leqslant|x-y|^p, \qquad 0<p<1,\\ &|x^p-y^p|<p|x-y|(\max\{|x|,|y|\})^{p-1}, \qquad p>1.\end{aligned}$$

Since $X_t=Y_t^{-1/(\beta-1)}$, we have

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}|X_t-(Y_t^n)^{-1/(\beta-1)}|&=\sup_{0\leqslant t\leqslant T}|Y_t^{-1/(\beta-1)}-(Y_t^n)^{-1/(\beta-1)}|\\ &=\sup_{0\leqslant t\leqslant T}\frac{|Y_t^{1/(\beta-1)}-(Y_t^n)^{1/(\beta-1)}|}{Y_t^{1/(\beta-1)}(Y_t^n)^{1/(\beta-1)}}.\end{aligned}$$

Note that

$$\inf_{0\leqslant t\leqslant T}Y_t^n\geqslant\min_{0\leqslant k\leqslant n}Y_{n,k}>0.$$

Thus

$$\sup_{0\leqslant t\leqslant T}|X_t-(Y_t^n)^{-1/(\beta-1)}|\leqslant\begin{cases}\sup_{0\leqslant t\leqslant T}\frac{|Y_t-\left(Y_t^n\right)|^{1/(\beta-1)}}{Y_t^{1/(\beta-1)}\left(Y_t^n\right)^{1/(\beta-1)}},\\ \text{if } \beta>2;\\ \sup_{0\leqslant t\leqslant T}\frac{|Y_t-Y_t^n|}{Y_tY_t^n},\\ \text{if } \beta=2;\\ \frac{1}{\beta-1}\sup_{0\leqslant t\leqslant T}\frac{|Y_t-Y_t^n|}{Y_t^{1/(\beta-1)}\left(Y_t^n\right)^{1/(\beta-1)}}\\ \times\left(\max\left\{\sup_{0\leqslant t\leqslant T}|Y_t|,\right.\right.\\ \left.\left.\sup_{0\leqslant t\leqslant T}|Y_t^n|\right\}\right)^{\frac{2-\beta}{\beta-1}},\\ \text{if } 1<\beta<2.\end{cases}$$

From (3.6) and the finiteness of $\sup_{0\leq t\leq T}|Y_t|$ we have

$$\begin{aligned}\sup_{0\leq t\leq T}|Y_t^n|&\leq\sup_{0\leq t\leq T}|Y_t^n-Y_t|+\sup_{0\leq t\leq T}|Y_t|\\&\leq O_\omega(n^{-H}\sqrt{\ln n})+\sup_{0\leq t\leq T}|Y_t|=O_\omega(1).\end{aligned}$$

This finishes the proof of (3.7).

3.3 Further investigations

In this section, we apply Proposition 5 to both the fractional AS model and the Heston-3/2 volatility model, illustrating the positive trajectories of these models. Later, we address the constraints associated with extending our findings to the CKLS model.

3.3.1 Ait-Sahalia model

The Ait-Sahalia-type SDE has the form

$$X_t=x_0+\int_0^t(a_1X_s^{-1}-a_2+a_3X_s-a_4X_s^r)\,ds+\sigma\int_0^t X_s^\beta\,dB_s^H \tag{3.10}$$

with the initial value $x_0>0$ and $r>2\beta-1$, where $H\in(\frac{1}{2},1)$, $\beta>1$, and deterministic constants $a_1,a_2,a_3,a_4>0$ and $\sigma>0$.

By using the Lamperti transformation $Y_t = X_t^{-(\beta-1)}$ we get

$$\begin{aligned}Y_t &= y_0 - (\beta-1)\int_0^t f(Y_s)\,ds - \sigma(\beta-1)B_t^H \\ &= y_0 + (\beta-1)\int_0^t \widehat{f}(Y_s)\,ds - (\beta-1)\sigma B_t^H,\end{aligned}$$

where $\widehat{f}(x) = -f(x)$ and

$$f(x) = a_1 x^{\frac{\beta+1}{\beta-1}} - a_2 x^{\frac{\beta}{\beta-1}} + a_3 x - a_4 x^{-\frac{r-\beta}{\beta-1}}.$$

The function f is continuously differentiable on $(0,+\infty)$. Condition $(\mathbf{C}_2)$ is satisfied since

$$\widehat{f}(x) - \frac{a_4}{2}x^{-\frac{r-\beta}{\beta-1}} = \frac{a_4}{2}x^{-\frac{r-\beta}{\beta-1}} - a_1 x^{\frac{\beta+1}{\beta-1}} + a_2 x^{\frac{\beta}{\beta-1}} - a_3 x \to \infty, \text{ as } x \to +0$$

with $a = a_4/2$, $\alpha = \frac{r-\beta}{\beta-1} - 1 > 0$.

Now we verify condition $(\mathbf{C}_3)$. Note that

$$\begin{aligned}\widehat{f}'(x) &= -a_1\frac{\beta+1}{\beta-1}x^{\frac{2}{\beta-1}} + a_2\frac{\beta}{\beta-1}x^{\frac{1}{\beta-1}} - a_3 - a_4\frac{r-\beta}{\beta-1}x^{-\frac{r-1}{\beta-1}} \\ &= -x^{-\frac{r-1}{\beta-1}}\left(a_4\frac{r-\beta}{\beta-1} + a_1\frac{\beta+1}{\beta-1}x^{\frac{1+r}{\beta-1}} - a_2\frac{\beta}{\beta-1}x^{\frac{r}{\beta-1}}\right) - a_3 \\ &= -x^{\frac{2}{\beta-1}}\left(a_1\frac{\beta+1}{\beta-1} - a_2\frac{\beta}{\beta-1}x^{-\frac{1}{\beta-1}} + a_4\frac{r-\beta}{\beta-1}x^{-\frac{r+1}{\beta-1}}\right) - a_3.\end{aligned}$$

Since the derivative $\widehat{f}'(x)$ is continuous on $(0,\infty)$, $\lim_{x\to 0^+}\widehat{f}'(x) = -\infty$, and $\lim_{x\to+\infty}\widehat{f}'(x) = -\infty$, there is a constant K such that $\widehat{f}'(x) \leqslant K$ for all $x \in (0,\infty)$.

Now the mean value theorem implies

$$(x-y)(\hat{f}(x) - \hat{f}(y)) = \hat{f}'(c)(x-y)^2 \leqslant K(x-y)^2, \tag{3.11}$$

where $c = x + \theta(y-x)$, $\theta \in (0,1)$. Thus equation (3.10) has a unique positive solution on $(0,+\infty)$.

Let us verify condition $(\mathbf{C}_4)$. Note that

$$\begin{aligned}\widehat{F}(x) =& x-(\beta-1)\widehat{f}(x)h \\ =& x-(\beta-1)\Big(-a_1x^{\frac{\beta+1}{\beta-1}}+a_2x^{\frac{\beta}{\beta-1}}-a_3x+a_4x^{-\frac{r-\beta}{\beta-1}}\Big)h \\ =& (\beta-1)\Big(a_1x^{\frac{\beta+1}{\beta-1}}-a_2x^{\frac{\beta}{\beta-1}}-a_4x^{-\frac{r-\beta}{\beta-1}}\Big)h+(1+a_3(\beta-1)h)x\end{aligned}$$

is continuous on $(0,\infty)$. It is clear that $1+a_3(\beta-1)h>0$ for any $h>0$ and

$$\lim_{x\to 0+}\widehat{F}(x)=-\infty \quad \text{and} \quad \lim_{x\to+\infty}\widehat{F}(x)=+\infty.$$

Since the function $\widehat{f}$ satisfies condition $(\mathbf{C}_3)$, condition $(\mathbf{C}_4)$ is satisfied as well. Therefore the conditions of Theorem 8 are satisfied.

Now, we can apply the Lamperti transform and approximation scheme (3.5) to simulate trajectories of the Ait-Sahalia model:

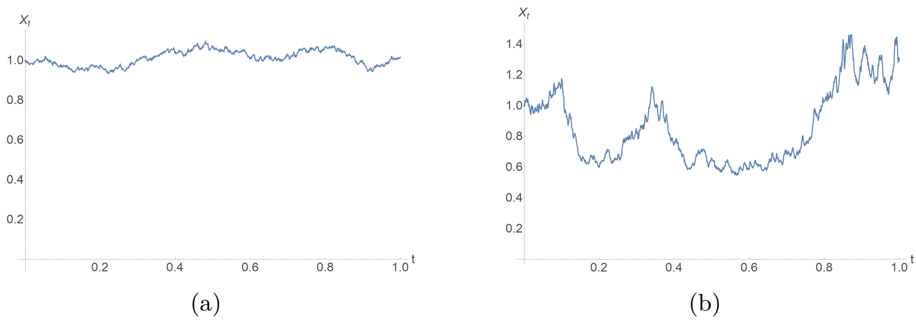


Figure 3.1: Trajectory of X in Eq. 3.10 ($a_1=a_2=a_3=a_4=1, \beta=1.5, H=0.55, r=2\beta$) using approximation scheme (3.5): (**a**) with $\sigma=0.2$. (**b**) with $\sigma=1$.

3.3.2 Heston volatility model

Consider the SDE

$$X_t=x_0+\int_0^t a_1X_s(a_2-X_s)\,ds+\sigma\int_0^t X_s^\beta\,dB_s^H \tag{3.12}$$

with the initial value $x_0>0$, where $H\in(\frac{1}{2},1)$, $1.5<\beta<2$, and deterministic constants $a_1,a_2,\sigma>0$. If $\beta=3/2$, then we call the SDE (3.12) the fractional Heston-3/2 volatility model, and additional limitation $a_2\in(0,1)$ has to be imposed.

By using the Lamperti transformation $Y_t = X_t^{-(\beta-1)}$ we get

$$\begin{aligned} Y_t &= y_0 - (\beta-1)\int_0^t f(Y_s)\,ds - \sigma(\beta-1)B_t^H \\ &= y_0 + (\beta-1)\int_0^t \widehat{f}(Y_s)\,ds - \sigma(\beta-1)B_t^H, \end{aligned}$$

where $\widehat{f}(x) = -f(x)$ and

$$f(x) = a_1 a_2 x - a_1 x^{-\frac{2-\beta}{\beta-1}}.$$

The function f is continuously differentiable on $(0,+\infty)$. Condition $(\mathbf{C}_2)$ is satisfied since

$$\widehat{f}(x) - \frac{a_1}{2}x^{-\frac{2-\beta}{\beta-1}} = \frac{a_1}{2}x^{-\frac{2-\beta}{\beta-1}} - a_1 a_2 x \to \infty \qquad \text{as } x \to +0$$

with $a = a_1/2$, $\alpha = \frac{2-\beta}{\beta-1} - 1 \geq 0$. Now we verify condition $(\mathbf{C}_3)$. Note that

$$\widehat{f}'(x) = -a_1 a_2 - a_1 \frac{2-\beta}{\beta-1} x^{-\frac{1}{\beta-1}} < -a_1 a_2.$$

Applying inequality (3.11), we obtain condition $(\mathbf{C}_3)$. Thus equation (3.12) has a unique positive solution on $(0,+\infty)$. From the obtained result it follows that the Heston-3/2 volatility model has a unique positive solution on $(0,+\infty)$.

Let us verify condition $(\mathbf{C}_4)$. Note that

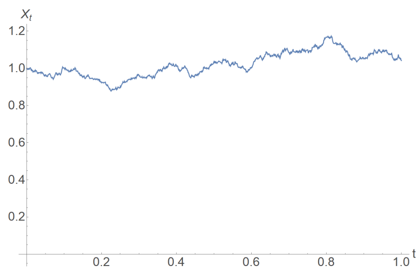
$$\begin{aligned} \widehat{F}(x) =& x - (\beta-1)\widehat{f}(x)h = x - (\beta-1)\Big(-a_1 a_2 x + a_1 x^{-\frac{2-\beta}{\beta-1}}\Big)h \\ =& (1 + a_1 a_2(\beta-1)h)x - (\beta-1)a_1 x^{-\frac{2-\beta}{\beta-1}} h \end{aligned}$$

is continuous on $(0,\infty)$. It is clear that $1 + a_3(\beta-1)h > 0$ for any $h > 0$ and

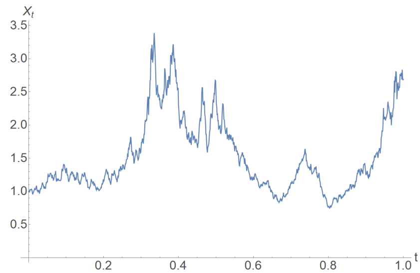
$$\lim_{x\to 0^+} \widehat{F}(x) = -\infty \quad \text{and} \quad \lim_{x\to+\infty} \widehat{F}(x) = +\infty.$$

Since the function $\widehat{f}$ satisfies condition $(\mathbf{C}_3)$, condition $(\mathbf{C}_4)$ is satisfied as well. Thus the conditions of Theorem 8 are satisfied.

Therefore, we can apply the Lamperti transform and approximation scheme (3.5) to simulate trajectories of Heston volatility model:



(a)



(b)

Figure 3.2: Trajectory of X in Eq. 3.12 ($a_1 = a_2 = 1, \beta = 1.75, H = 0.55$) using approximation scheme in (3.5): (**a**) with $\sigma = 0.2$. (**b**) with $\sigma = 1$.

3.3.3 CKLS model

Unfortunately, our proof scheme provides no insights into the trajectories' behaviour for the CKLS model

$$X_t = x_0 + \int_0^t (a_1 - a_2 X_s)\, ds + \sigma \int_0^t X_s^\beta\, dB_s^H, \qquad \beta > 1, \quad H \in (\tfrac{1}{2}, 1), \tag{3.13}$$

with the initial value $x_0 > 0$ and deterministic constants $a_1 > 0$, $a_2 \in \mathbb{R}$, and $\sigma > 0$.

Consider the SDE

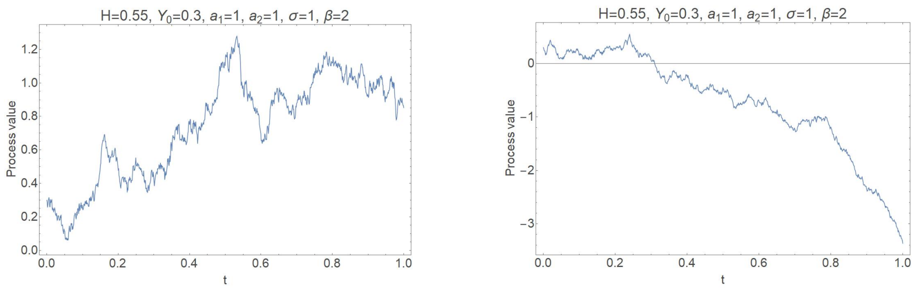
$$Y_t = y_0 + (\beta - 1) \int_0^t (a_2 Y_s - a_1 Y_s^{\beta/(\beta-1)}) ds - (\beta - 1)\sigma B_t^H,\ H \in (\tfrac{1}{2}, 1). \tag{3.14}$$

Suppose that the solution of the SDE (3.14) is positive. Then by applying the chain rule (see Theorem 6) and the inverse Lamperti transform $X_t = Y_t^{-1/(\beta-1)}$ we can prove that X is a positive solution of (3.13).

It is easy to see that the function $\widehat{f}(x) = a_2 x - a_1 x^{\beta/(\beta-1)}$ does not satisfy condition ($\mathbf{C}_2$). So we cannot apply Proposition 5 and say anything about the positivity of the solution of (3.14).

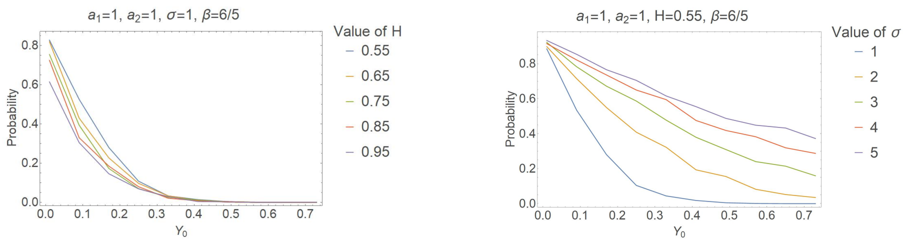
Computer modelling shows that the trajectories of the process Y may have negative values for $y_0 > 0$ (see Figure 3.3).

To investigate the probability of reaching the negative values by the process Y when $t \in [0, 1]$, we simulate the *exact* solution by using the backward Euler approximation scheme for step size $h = 10^{-3}$ and repeat this process 10^3 times counting the trajectories with negative values. We observe that the solution has a higher probability of reaching the negative values for small initial values Y_0 and that for large enough values of Y_0, this probability tends to zero. Additionally, the probability increases for greater values of the parameters σ, β (see Figures 3.4**b** and 3.6) and decreases for greater values of the parameters H, a_2 (see Figures 3.4**a** and 3.5**b**). The influence of a_1 on the probability (see Figure 3.5**a**) is not noticeable in comparison with other parameters.



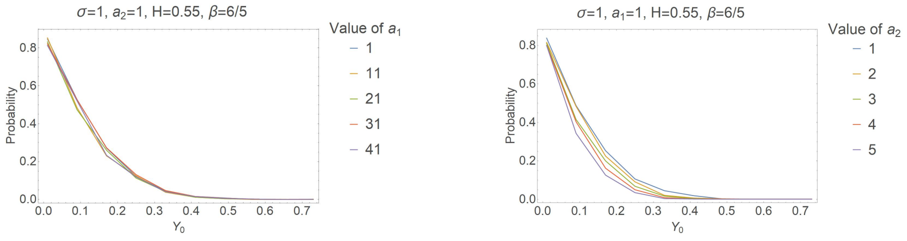
(a) (b)

Figure 3.3: Trajectory of Y: (**a**) with negative values. (**b**) with positive values.



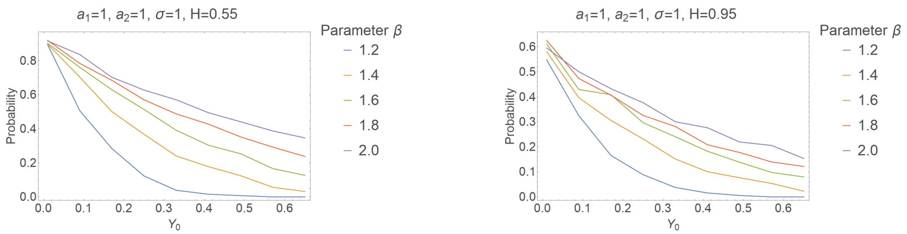
(a) (b)

Figure 3.4: Probability that the trajectories have negative values:
(**a**) Dependence of probability on Y_0 for different H. (**b**) Dependence of probability on Y_0 for different σ.



(a) (b)

Figure 3.5: Probability that the trajectories have negative values: (**a**) Dependence of probability on a_1. (**b**) Dependence of probability on a_2.



(a) (b)

Figure 3.6: Probability that the trajectories have negative values: (**a**) Dependence of probability on β when $H = 0.55$. (**b**) Dependence of probability on β when $H = 0.95$.

Thus we can only state that equation (3.13) has a solution $X_t = Y_t^{-1/(\beta-1)}$ until the moment at which Y becomes zero. On the other hand, we do know that the CKLS model driven by a standard Brownian motion (see [57]) with $\beta > 1$ and the fractional CKLS model with $1/2 \leqslant \beta < 1$ (see [34]) have positive solutions.

3.3.4 Hurst index estimates

For positive solutions of equation (3.3), we construct a strongly consistent and asymptotically normal estimator of the Hurst parameter H from discrete observations of a single sample path.

For a real-valued process $X = \{X_t, t \in [0,T]\}$, we define the second order increments along uniform partitions as

$$\Delta^{(2)}_{n,k} X = X_{t^n_{k+1}} - 2X_{t^n_k} + X_{t^n_{k-1}}, \quad 1 \leqslant k \leqslant n-1.$$

Theorem 9. *Let X be a unique positive solution of SDE* (3.3) *with $H \in (\frac{1}{2}, 1)$. Then*

$$\widehat{H}_n = H + O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right)$$

and

$$2\ln 2\sqrt{n}\,(\widehat{H}_n - H) \xrightarrow{d} \mathcal{N}(0, \sigma_H^2)$$

with known variance σ_H^2 defined in Appendix A, where

$$\widehat{H}_n = \frac{1}{2} - \frac{1}{2\ln 2}\ln\left(\frac{\widetilde{V}^{(2)X}_{2n,T}}{\widetilde{V}^{(2)X}_{n,T}}\right), \qquad \widetilde{V}^{(2)X}_{n,T} = \sum_{k=1}^{n-1}\left(\frac{\Delta^{(2)}_{n,k}X}{X^{\beta}_{t^n_k}}\right)^2.$$

The proof of this theorem follows the blueprint of Theorem 2 proof presented in [34] .

Chapter 4

Pathwise convergent approximation for the fractional SDEs

In this chapter, we shall consider a one-dimensional SDE driven by an arbitrary stochastic process $Z=(Z_t)_{t\geqslant 0}$, $Z_0=0$, with Hölder continuous paths of order $1/2<\gamma<1$

$$X_t = x_0 + \int_0^t \alpha(X_s)\,ds + \int_0^t \sigma(X_s)\,dZ_s, \qquad x_0\in\mathbb{R}, \quad t\in[0,T]. \tag{4.1}$$

The whole equation is understood as a pathwise Riemann–Stieltjes integral equation again. As a special case, we can take $Z=B^H$, where B^H is a fractional Brownian motion (fBm).

We aim to explore the numerical approximation of the SDE (6.6), taking into account extra assumptions regarding the diffusion coefficient. We will require the diffusion term to be strictly positive, that is, $\inf_{x\in\mathbb{R}}\sigma(x)>0$. As the diffusion coefficient is strictly positive, we can apply the Lamperti transformation, which allows us to convert a complex SDE into a simpler one with a constant diffusion coefficient. By approximating the transformed SDE using a selected scheme, we can then obtain an approximation scheme for the initial SDE through inverse transformation.

4.1 Main results

Assume that σ is continuously differentiable, and for some $L>0$ and for any $x,y\in\mathbb{R}$,

$$|\alpha(x)|+|\sigma(x)|\leqslant L(1+|x|), \qquad |\sigma'(x)|\leqslant L, \tag{4.2}$$

$$|\alpha(x)-\alpha(y)|+|\sigma'(x)-\sigma'(y)|\leqslant L|x-y|, \tag{4.3}$$

and the process Z has Hölder continuous paths of order $1/2 < \gamma < 1$, i.e., there exists a random variable $G_{\gamma,T} = G_{\gamma,T}(\omega) \in (0,\infty)$ such that

$$|Z_t - Z_s| \leqslant G_{\gamma,T}|t-s|^\gamma, \quad s,t \in [0,T].$$

Definition 11. *Let $1/2 < \gamma < 1$, $\alpha \in (1-\gamma, 1/2)$. By $W_\infty^\alpha([0,T])$ denote the space of real-valued measurable functions $f\colon [0,T] \to \mathbb{R}$ such that*

$$\|f\|_{\infty,\alpha;T} = \sup_{s\in[0,T]} \left(|f(s)| + \int_0^s |f(s)-f(u)|(s-u)^{-1-\alpha} du \right) < \infty.$$

Under these conditions given in [45] (see also [48]) Eq. (4.1) has a unique solution X such that $\|X\|_{\alpha,\infty,T} < \infty$ a.s. for any $\alpha \in (1-\gamma, \frac{1}{2})$.

In addition to the conditions formulated in (4.2)-(4.3), we will require that the diffusion term be such that

(**H**) $\inf_{x\in\mathbb{R}} \sigma(x) > 0$.

Note that condition (4.3) implies that the function $1/\sigma(x)$ is continuously differentiable on $\mathbb{R}$. Thus under condition (**H**) the Lamperti transform

$$F(x) = \int_0^x \frac{1}{\sigma(y)}\, dy, \qquad x \in \mathbb{R}$$

has the inverse function $F^{-1} : \mathbb{R} \to \mathbb{R}$ which is strictly monotone and differentiable

$$(F^{-1})'(x) = \sigma(F^{-1}(x)), \qquad x \in \mathbb{R}. \tag{4.4}$$

Set $Y_t = F(X_t)$. By chain rule we obtain

$$Y_t = Y_0 + \int_0^t F'(X_s)\, dX_s = Y_0 + \int_0^t \frac{\alpha(X_s)}{\sigma(X_s)}\, ds + Z_t = y_0 + \int_0^t f(Y_s)\, ds + Z_t,$$

where $y_0 = F(x_0)$,

$$f(x) = \widehat{f}(F^{-1}(x)), \qquad \widehat{f}(x) = \frac{\alpha(x)}{\sigma(x)}.$$

Since under conditions (4.2)-(4.3) there exists a unique solution of (4.1), the equation

$$Y_t = y_0 + \int_0^t f(Y_s)\, ds + Z_t \tag{4.5}$$

has a unique solution under these same conditions.

4.1.1 Conditions

To state our main results, we use the following requirements on function f:

($\mathbf{C}_0$) f is continuously differentiable on $\mathbb{R}$;

($\mathbf{C}_1$) Assume that there exists a constant $K \in \mathbb{R}$ such that $f'(x) \leqslant K$ for all $x \in \mathbb{R}$;

($\mathbf{C}_2$) Assume that there exists a constant $M \geqslant 0$ such that $g'(x) \geqslant -M$ for all $x \in \mathbb{R}$, where $g(x) = f(x)f'(x)$;

($\mathbf{C}_3$) Assume that the function f on $\mathbb{R}$ is twice continuous differentiable and there exists a constant $N \geqslant 0$ such that $|f''(x)| \leqslant N$ for all $x \in \mathbb{R}$.

4.1.2 Theorems

Let $\pi = \{t_k^n = \frac{k}{n}T, 1 \leqslant k \leqslant n\}$ be a sequence of uniform partitions of the interval $[0,T]$, and let $h = t_k^n - t_{k-1}^n$, $1 \leqslant k \leqslant n$. For the solution Y of the SDE (4.5) define the following backward approximation scheme:

$$\begin{aligned}
&Y_{n,k+1} - f(Y_{n,k+1})h + f'(Y_{n,k+1})f(Y_{n,k+1})\frac{h^2}{2} \\
&\quad = Y_{n,k} + (Z_{t_{k+1}^n} - Z_{t_k^n}) - f'(Y_{n,k})\int_{t_k^n}^{t_{k+1}^n}(Z_{t_{k+1}^n} - Z_s)ds, \qquad \text{(A)} \\
&\quad Y_{n,0} = y_0,\ 0 \leqslant k \leqslant n-1.
\end{aligned}$$

Define

$$h_0 := \frac{\sqrt{2M + (K^+)^2} - K^+}{M}, \tag{4.6}$$

where constants K and M are given in ($\mathbf{C}_1$) and ($\mathbf{C}_2$), $K^+ = \max\{0, K\}$.

Theorem 10. *Suppose that the function f in (4.5) satisfies conditions ($\boldsymbol{C}_0$) – ($\boldsymbol{C}_3$). Assume that the sequence of uniform partitions π of the interval $[0,T]$ is such that $h < h_0$. Then for $\gamma \in (\frac{1}{2}, 1)$ it follows*

$$\max_{1 \leqslant k \leqslant n} |Y_{t_k^n} - Y_{n,k}| = O_\omega(h^{2\gamma}).$$

Remark 3. *Note that this result is not applicable for CKLS, Heston-3/2 volatility, and Ait-Sahalia models, since condition ($\boldsymbol{C}_3$) is not satisfied.*

Theorem 11. *Assume that SDE (4.1) has unique solution and conditions of Theorem 10 are satisfied. Then for $\gamma \in (\frac{1}{2}, 1)$ it follows that*

$$\max_{1 \leqslant k \leqslant n} |X_{t_k^n} - F^{-1}(Y_{n,k})| = O_\omega(h^{2\gamma}).$$

4.2 Proofs

First, we show that the backward approximation (A) is well defined.

Lemma 1. *Let the conditions* ($\boldsymbol{C}_1$) *and* ($\boldsymbol{C}_2$) *are satisfied. The function*

$$H(x) = x - f(x)h + f'(x)f(x)\frac{h^2}{2}, \qquad x \in \mathbb{R},$$

is strictly monotone and $\lim_{x \to +\infty} H(x) = +\infty$, $\lim_{x \to -\infty} H(x) = -\infty$ *for any* $h < h_0$.

Remark 4. *Note that for* $K \leqslant 0$ *and* $g'(x) \geqslant 0$ *there is no restriction on* h.

Proof. Under the assumptions ($\mathbf{C}_1$) and ($\mathbf{C}_2$) the function $H(x)$ is strictly monotone. Indeed,

$$\begin{aligned}(x-y)(H(x)-H(y)) =& (x-y)^2 - (x-y)(f(x)-f(y))h \\ &+ (x-y)(f'(x)f(x) - f'(y)f(y))\frac{h^2}{2} \\ \geqslant& \Big(1 - K^+ h - M\frac{h^2}{2}\Big)(x-y)^2 > 0 \end{aligned} \tag{4.7}$$

for $h < h_0$.

Now we find the limits of the function $H(x)$ as $x \to \pm\infty$. From (4.7) it follows that

$$H(x) \leqslant H(x_0) + \Big(1 - K^+ h - M\frac{h^2}{2}\Big)(x - x_0)$$

for $x < x_0$, which gives $\lim_{x \to -\infty} H(x) = -\infty$. Inequality (4.7) implies

$$H(x) \geqslant H(x_0) + \Big(1 - K^+ h - M\frac{h^2}{2}\Big)(x - x_0)$$

for $x > x_0$ and thus $\lim_{x \to \infty} H(x) = \infty$. ∎

From Lemma 1 it follows that for each $b \in \mathbb{R}$ the equation $H(x) = b$ has a unique solution for $0 < h < h_0$. Consequently, the backward approximation scheme is well defined if conditions $(\mathbf{C}_1)$ and $(\mathbf{C}_2)$ are satisfied and $h < h_0$.

4.2.1 Proof of Theorem 10

Applying the chain rule we obtain that

$$\begin{aligned}
&f(Y_{t^n_{k+1}}) - f(Y_s) \\
&\quad = \int_s^{t^n_{k+1}} f'(Y_u)\, dY_u = \int_s^{t^n_{k+1}} f'(Y_u) f(Y_u)\, du + \int_s^{t^n_{k+1}} f'(Y_u)\, dZ_u \\
&\quad = -\int_s^{t^n_{k+1}} [f'(Y_{t^n_{k+1}}) f(Y_{t^n_{k+1}}) - f'(Y_u) f(Y_u)] du \\
&\qquad + f'(Y_{t^n_{k+1}}) f(Y_{t^n_{k+1}})(t^n_{k+1} - s) \\
&\qquad + \int_s^{t^n_{k+1}} [f'(Y_u) - f'(Y_s)]\, dZ_u + [f'(Y_s) - f'(Y_{t^n_k})](Z_{t^n_{k+1}} - Z_s) \\
&\qquad + f'(Y_{t^n_k})(Z_{t^n_{k+1}} - Z_s).
\end{aligned} \tag{4.8}$$

From Eq. (4.5) we get that

$$\begin{aligned}
Y_{t^n_{k+1}} =& Y_{t^n_k} + f(Y_{t^n_{k+1}}) h - \int_{t^n_k}^{t^n_{k+1}} [f(Y_{t^n_{k+1}}) - f(Y_s)]\, ds + (Z_{t^n_{k+1}} - Z_{t^n_k}) \\
=& Y_{t^n_k} + f(Y_{t^n_{k+1}}) h + (Z_{t^n_{k+1}} - Z_{t^n_k}) \\
& - f'(Y_{t^n_{k+1}}) f(Y_{t^n_{k+1}}) \int_{t^n_k}^{t^n_{k+1}} (t^n_{k+1} - s)\, ds \\
& - f'(Y_{t^n_k}) \int_{t^n_k}^{t^n_{k+1}} [Z_{t^n_{k+1}} - Z_s] ds + R_{n,k+1} \\
=& Y_{t^n_k} + f(Y_{t^n_{k+1}}) h + (Z_{t^n_{k+1}} - Z_{t^n_k}) - f'(Y_{t^n_k+1}) f(Y_{t^n_k+1}) \frac{h^2}{2} \\
& - f'(Y_{t^n_k}) \int_{t^n_k}^{t^n_{k+1}} [Z_{t^n_{k+1}} - Z_s] ds + R_{n,k+1},
\end{aligned}$$

where

$$R_{n,k+1} = \int_{t_k^n}^{t_{k+1}^n} \int_s^{t_{k+1}^n} [f'(Y_{t_{k+1}^n}) f(Y_{t_{k+1}^n}) - f'(Y_u) f(Y_u)]\, du ds$$
$$- \int_{t_k^n}^{t_{k+1}^n} \int_s^{t_{k+1}^n} [f'(Y_u) - f'(Y_s)]\, dZ_u ds$$
$$- \int_{t_k^n}^{t_{k+1}^n} [f'(Y_s) - f'(Y_{t_k^n})](Z_{t_{k+1}^n} - Z_s)\, ds \tag{4.9}$$

is the remainder term.

For simplicity of notation, we introduce the following

$$\begin{aligned}
\zeta_{n,k+1} =& f'(Y_{t_{k+1}^n} + \theta(Y_{n,k+1} - Y_{t_{k+1}^n})), \quad \text{where } \theta = \theta_{n,k+1} \in (0,1),\\
\eta_{n,k} =& f''(Y_{t_k^n} + \vartheta(Y_{n,k} - Y_{t_k^n})), \qquad\qquad \text{where } \vartheta = \vartheta_{n,k} \in (0,1),\\
\rho_{n,k+1} =& g'(Y_{t_{k+1}^n} + \kappa(Y_{n,k+1} - Y_{t_{k+1}^n})), \quad \text{where } \kappa = \kappa_{n,k+1} \in (0,1)
\end{aligned}$$

and $g(x) = f'(x) f(x)$. Then the difference of Eq. (4.8) and approximation (A) is

$$\begin{aligned}
&Y_{t_{k+1}^n} - Y_{n,k+1}\\
&\quad = Y_{t_k^n} - Y_{n,k} + h(f(Y_{t_{k+1}^n}) - f(Y_{n,k+1})) - (f'(Y_{t_{k+1}^n}) f(Y_{t_{k+1}^n})\\
&\qquad - f'(Y_{n,k+1}) f(Y_{n,k+1}))\frac{h^2}{2} - (f'(Y_{t_k^n})\\
&\qquad - f'(Y_{n,k})) \int_{t_k^n}^{t_{k+1}^n} [Z_{t_{k+1}^n} - Z_s] ds + R_{n,k+1}\\
&\quad = Y_{t_k^n} - Y_{n,k} + \zeta_{n,k+1}(Y_{t_{k+1}^n} - Y_{n,k+1})h - \rho_{n,k+1}(Y_{t_{k+1}^n} - Y_{n,k+1})\frac{h^2}{2}\\
&\qquad - \eta_{n,k}(Y_{t_k^n} - Y_{n,k}) \int_{t_k^n}^{t_{k+1}^n} [Z_{t_{k+1}^n} - Z_s] ds + R_{n,k+1}
\end{aligned}$$

and

$$\begin{aligned}
(Y_{t_{k+1}^n} - Y_{n,k+1})&\left[1 - \zeta_{n,k+1} h + \rho_{n,k+1}\frac{h^2}{2}\right]\\
&= (Y_{t_k^n} - Y_{n,k})[1 - \lambda_{n,k}] + R_{n,k+1},
\end{aligned} \tag{4.10}$$

where

$$\lambda_{n,k+1} = \eta_{n,k} \int_{t_k^n}^{t_{k+1}^n} (Z_{t_{k+1}^n} - Z_s) ds.$$

Applying assumptions $(\mathbf{C}_1)$ and $(\mathbf{C}_2)$ we transform equality (4.10) into a recursive inequality. Indeed,

$$1-\zeta_{n,k+1}h+\rho_{n,k+1}\frac{h^2}{2}\geqslant 1-K^+h-M\frac{h^2}{2}>0$$

for $h<h_0$. Thus,

$$(Y_{t_{k+1}^n}-Y_{n,k+1})\Big(1-K^+h-M\frac{h^2}{2}\Big)\leqslant (Y_{t_k^n}-Y_{n,k})(1-\lambda_{n,k+1})+R_{n,k+1}.$$

Further, by applying inequality $1+x\leqslant e^x$, $\forall x\geqslant 0$, we get that

$$|y_{n,k+1}|\leqslant |y_{n,k}|\exp\{|\lambda_{n,k+1}|\}\varepsilon_h^{-1}+|R_{n,k+1}|\varepsilon_h^{-1}, \tag{4.11}$$

where $y_{n,k}=Y_{t_k^n}-Y_{n,k}$ and $\varepsilon_h=1-K^+h-M\,\frac{h^2}{2}$.

Now, recursively from (4.11) we get that

$$|y_{n,k+1}|\leqslant \sum_{j=0}^{k}\varepsilon_h^{-(k+1-j)}\exp\Big\{\sum_{i=j+1}^{k}|\lambda_{n,i+1}|\Big\}|R_{n,j+1}|.$$

We have $\ln\frac{1}{1-x}\leqslant \frac{x}{1-x}$, $0\leqslant x<1$, then we get that

$$\begin{aligned}\varepsilon_h^{-(k+1-j)}&\leqslant \exp\Big(n\ln\frac{1}{\varepsilon_h}\Big)\leqslant \exp\Bigg(n\,\frac{K^+h+M\,\frac{h^2}{2}}{\varepsilon_h}\Bigg)\\&\leqslant \exp\Big(\frac{K^+T+MT}{\varepsilon_h}\Big),\end{aligned}$$

if $h\leqslant 2$. Consequently,

$$\begin{aligned}|y_{n,k+1}|&\leqslant \exp\Big(\frac{K^+T+MT}{\varepsilon_h}\Big)\sum_{j=0}^{k}\exp\Big\{\sum_{i=j+1}^{k}|\lambda_{n,i+1}|\Big\}|R_{n,j+1}|\\&\leqslant \psi_n\sum_{j=0}^{k}|R_{n,j+1}|,\end{aligned}$$

where

$$\psi_n=\exp\Big(\frac{K^+T+MT}{\varepsilon_h}\Big)\exp\Big\{n\max_{1\leqslant i\leqslant n}|\lambda_{n,i}|\Big\}.$$

To finish the proof of Theorem 10 it remains to estimate $\max_{1\leqslant j\leqslant n}|R_{n,j}|$ and $\max_{1\leqslant i\leqslant n}|\lambda_{n,i}|$. From Lemmas 2 and 3 below it follows that

$$\psi_n = \exp\left(\frac{K^+T + MT}{\varepsilon_h}\right)\exp\{nO_\omega(h^{1+\gamma})\} = O_\omega(1).$$

Thus,

$$\max_{1\leqslant k\leqslant n}|y_{n,k}| = O_\omega(h^{2\gamma}).$$

Lemma 2. *Under the assumptions of Theorem 10*

$$\max_{1\leqslant j\leqslant n}|R_{n,j}| = O_\omega(h^{1+2\gamma}).$$

Proof. We start by estimating the first term $R_{n,j}$ in (4.9). Since the function $g'(x)$ is continuous and the process Y is continuous then $\sup_{0\leqslant t\leqslant T}|g'(Y_t)| < \infty$. From Hölder continuity of Z we get

$$\begin{aligned}|Y_t - Y_s| \leqslant &\int_s^t |f(Y_u)|\,du + |Z_t - Z_s|\\ \leqslant &(t-s)\sup_{0\leqslant u\leqslant T}|f(Y_u)| + G_{\gamma,T}|t-s|^\gamma = O_\omega(|t-s|^\gamma).\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{t_k^n}^{t_{k+1}^n}\int_s^{t_{k+1}^n}|g(Y_u) - g(Y_s)|\,duds\\ &\quad\leqslant \sup_{0\leqslant t\leqslant T}|g'(Y_s)| \sup_{t_k^n\leqslant s\leqslant t_{k+1}^n}\ \sup_{s\leqslant u\leqslant t_{k+1}^n}|Y_u - Y_s|h^2 = O_\omega(h^{2+\gamma}).\end{aligned}$$

From the Love-Young inequality (see Theorem 4), it follows that

$$\begin{aligned}&\left|\int_s^{t_{k+1}^n}[f'(Y_u) - f'(Y_s)]\,dZ_u\right|\\ &\qquad\leqslant \zeta(2\gamma)|Y|_\gamma|Z|_\gamma \sup_{0\leqslant t\leqslant T}|f''(Y_t)|h^{2\gamma} = O_\omega(h^{2\gamma}).\end{aligned} \tag{4.12}$$

Therefore, the second term $R_{n,j}$ in (4.9) has the following estimate

$$\begin{aligned}&\left|\int_{t_k^n}^{t_{k+1}^n}\int_s^{t_{k+1}^n}[f'(Y_u) - f'(Y_s)]\,dZ_u ds\right|\\ &\qquad\leqslant \int_{t_k^n}^{t_{k+1}^n}\left|\int_s^{t_{k+1}^n}[f'(Y_u) - f'(Y_s)]\,dZ_u\right|ds = O_\omega(h^{1+2\gamma}).\end{aligned}$$

Finally, for the third term of $R_{n,j}$ in (4.9) we obtain the estimate

$$\int_{t_k^n}^{t_{k+1}^n} |f'(Y_s) - f'(Y_{t_k^n})| \cdot |Z_{t_{k+1}^n} - Z_s|\, ds$$
$$\leqslant \sup_{0 \leqslant t \leqslant T} |f''(Y_t)| \int_{t_k^n}^{t_{k+1}^n} |Y_s - Y_{t_k^n}| \cdot |Z_{t_{k+1}^n} - Z_s|\, ds = O_\omega(h^{1+2\gamma}).$$

The proof is complete. □

Lemma 3. *Assume that condition* ($\boldsymbol{C}_3$) *is satisfied. Then*

$$\max_{1 \leqslant k \leqslant n} |\lambda_{n,k}| = O_\omega(h^{1+\gamma}).$$

Proof. Applying condition ($\mathbf{C}_3$) we get

$$|\eta_{n,k}| \leqslant N.$$

Since

$$\int_{t_k^n}^{t_{k+1}^n} |Z_{t_{k+1}^n} - Z_s|\, ds = O_\omega(h^{1+\gamma})$$

then the proof is complete. □

4.2.2 Proof of Theorem 11.

Note that

$$\begin{aligned} |X_{t_k^n} - F^{-1}(Y_{n,k})| &= |F^{-1}(Y_{t_k^n}) - F^{-1}(Y_{n,k})| \\ &= |(F^{-1}(Y_{t_k^n} + \theta_{n,k}(Y_{t_k^n} - Y_{n,k})))'| \cdot |Y_{t_k^n} - Y_{n,k}| \\ &= |\sigma(F^{-1}(Y_{t_k^n} + \theta_{n,k}(Y_{t_k^n} - Y_{n,k})))| \cdot |Y_{t_k^n} - Y_{n,k}| \end{aligned} \tag{4.13}$$

where $\theta = \theta_{n,k} \in (0,1)$, and

$$\max_{1 \leqslant k \leqslant n} |Y_{t_k^n} + \theta_{n,k}(Y_{t_k^n} - Y_{n,k})| \leqslant \sup_{0 \leqslant t \leqslant T} |Y_t| + O_\omega(h^{2\gamma}) = O_\omega(1). \tag{4.14}$$

Since the function $\sigma(x)$ satisfies condition (4.2), then

$$|\sigma(F^{-1}(x))| \leqslant L(1 + |(F^{-1}(x)|).$$

In addition, as the function $F^{-1}(x)$ increases and (4.14) is satisfied, then there exists a random variable $0 < \zeta < \infty$ such that

$$\max_{1\leqslant k\leqslant n} |F^{-1}(Y_{t_k^n} + \theta_{n,k}(Y_{t_k^n} - Y_{n,k}))| \leqslant \max\{|F^{-1}(-\zeta)|, F^{-1}(\zeta)\} = O_\omega(1).$$

Thus, the required result follows from Theorem 10.

4.3 Further investigations

In this section, we will employ the derived theoretical results for particular SDEs. Specifically, the Pearson fractional diffusion process and the model from [26] based on trigonometric functions, both of which fulfill the conditions of Theorem 10. We decided to explore the fractional Pearson diffusion process as it is more prominent and widely utilized. However, the same simulation and analysis methodology can be implemented for the second model too.

4.3.1 Pearson diffusion

Consider the Pearson diffusion process

$$dX_t = \alpha(X_t)\,dt + \sigma(X_t)\,dB_t^H \tag{4.15}$$

with

$$\alpha(x) = b - ax, \qquad \sigma(x) = \sqrt{\sigma_0 + \sigma_1 x + \sigma_2 x^2}.$$

Assume that $\inf_{x\in\mathbb{R}} \sigma(x) > 0$ and $a, b, \sigma \in \mathbb{R}$. Then the diffusion coefficient $\sigma(x)$ has bounded first and second-order derivatives and equation (4.15) has a unique solution since the drift and diffusion coefficients satisfy conditions (4.2)-(4.3).

Then equation (4.15) after Lamperti transform has form

$$Y_t = Y_0 + \int_0^t f(Y_s)\,ds + B_t^H, \tag{4.16}$$

where

$$f(x) = \widehat{f}(F^{-1}(x)), \qquad \widehat{f}(x) = \frac{\alpha(x)}{\sigma(x)}.$$

To check the conditions ($\mathbf{C}_0$)-($\mathbf{C}_3$) we have to find the expressions of the functions $f'(x)$, $(ff')'(x)$, $f''(x)$.

Note that

$$
\begin{aligned}
f'(x) =&(\widehat{f}(F^{-1}(x)))' = \widehat{f}'(F^{-1}(x))(F^{-1})'(x) = \widehat{f}'(F^{-1}(x))\sigma(F^{-1}(x)) \\
=&\Big(\alpha' - \frac{\alpha\sigma'}{\sigma}\Big)(F^{-1}(x)), \qquad (4.17)
\end{aligned}
$$

where

$$
\widehat{f}'(x) = \Big(\frac{\alpha'\sigma - \alpha\sigma'}{\sigma^2}\Big)(x).
$$

Since

$$
(\widehat{f}\widehat{f}')(x) = \Big(\frac{\alpha\alpha'\sigma - \alpha^2\sigma'}{\sigma^3}\Big)(x) = \Big(\frac{\alpha\alpha'}{\sigma^2} - \frac{\alpha^2\sigma'}{\sigma^3}\Big)(x)
$$

and

$$
\begin{aligned}
(\widehat{f}\widehat{f}')'(x) =&\frac{[(\alpha'(x))^2 + \alpha(x)\alpha''(x)]\sigma(x) - 2\alpha(x)\alpha'(x)\sigma'(x)}{\sigma^3(x)} \\
&- \frac{[2\alpha(x)\alpha'(x)\sigma'(x) + \alpha^2(x)\sigma''(x)]\sigma(x) - 3\alpha^2(x)(\sigma'(x))^2}{\sigma^4(x)},
\end{aligned}
$$

since $(F^{-1})'(x) = \sigma(F^{-1}(x))$ (see (4.4)), then

$$
\begin{aligned}
(ff')'(x) =&(\widehat{f}\widehat{f}')'(F^{-1}(x))\sigma(F^{-1}(x)) \\
=&\Big(\frac{[(\alpha')^2 + \alpha\alpha'']\sigma - 2\alpha\alpha'\sigma'}{\sigma^2}\Big)(F^{-1}(x)) \\
&- \Big(\frac{[2\alpha\alpha'\sigma' + \alpha^2\sigma'']\sigma - 3\alpha^2(\sigma')^2}{\sigma^3}\Big)(F^{-1}(x)).
\end{aligned}
$$

Further,

$$
\begin{aligned}
(\widehat{f}')'(x) =&\frac{\alpha''(x)\sigma(x) - \alpha'(x)\sigma'(x)}{\sigma^2(x)} \\
&- \frac{[\alpha'(x)\sigma'(x) + \alpha(x)\sigma''(x)]\sigma^2(x) - 2\alpha(x)\sigma(x)(\sigma'(x))^2}{\sigma^4(x)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f''(x) =&\widehat{f}''(F^{-1}(x))\sigma(F^{-1}(x)) \\
=&\Big(\frac{\alpha''\sigma - \alpha'\sigma'}{\sigma}\Big)(F^{-1}(x)) - \Big(\frac{[\alpha'\sigma' + \alpha\sigma'']\sigma^2 - 2\alpha\sigma(\sigma')^2}{\sigma^3}\Big)(F^{-1}(x)).
\end{aligned}
$$

To simplify the analysis of derivatives and to allow the simulation itself, we take specific expressions of coefficients. Let

$$\sigma(x)=\sqrt{x^2+2x+2}, \qquad a=1, \qquad b=2. \tag{4.18}$$

The Lamperti transformation of σ and its inverse are defined as follows

$$\begin{aligned} F(x) &= \int_0^x \frac{dy}{\sqrt{y^2+2y+2}} = \ln\left(\frac{x+1+\sqrt{(x+1)^2+1}}{1+\sqrt{2}}\right), \\ F^{-1}(x) &= \frac{\sqrt{2}+1}{2}e^x - \frac{\sqrt{2}-1}{2}e^{-x} - 1. \end{aligned}$$

Note that

$$(F^{-1})'(x) = \frac{\sqrt{2}+1}{2}e^x + \frac{\sqrt{2}-1}{2}e^{-x} \leqslant \left(\sqrt{2}+1\right)e^{|x|}.$$

Thus, from (4.14) it follows that

$$(F^{-1}(Y_{t_k^n}+\theta_{n,k}(Y_{t_k^n}-Y_{n,k})))' = O_\omega(1).$$

This is enough for Theorem 11 to be true (see equality (4.13)) if conditions of Theorem 10 are satisfied.

Now we verify conditions ($\mathbf{C}_0$)-($\mathbf{C}_3$). Condition ($\mathbf{C}_0$) is satisfied (see (4.17)). By inserting the expressions of coefficients a and b, we get

$$\begin{aligned} |\widehat{f}'(x)\sigma(x)| &= \left| -\frac{3x+4}{x^2+2x+2} \right| < 2.09 \\ |\widehat{f}''(x)\sigma(x)| &= \left| \frac{3(2x^2+5x+2)}{(x^2+2x+2)^2} \right| < 1.56 \\ \left| (\widehat{f}\widehat{f}')'(x)\sigma(x) \right| &= \left| \frac{-6x^3+6x^2+48x+28}{(x^2+2x+2)^{5/2}} \right| < 9.63. \end{aligned}$$

Since the function $F^{-1}(x)$ is continuous and increasing then from above it follows that the functions $f'(x)$, $f''(x)$, and $(ff')'(x)$ are bounded. Thus, conditions ($\mathbf{C}_1$)-($\mathbf{C}_3$) are satisfied.

4.3.2 Numerical simulation

Notice that the approximation Scheme (A) is not *final.* In order to get the original process X_t approximation, the inverse Lamperti transform has to be applied. However, Scheme (A) is implicit, and both the number of calculations needed and the general complexity of the approximation can be reduced by combining the Scheme (3.5) and Lamperti transform simultaneously:

$$
\begin{aligned}
&F(X_{n,k+1}) - \widehat{f}(X_{n,k+1})h + \widehat{f}'(X_{n,k+1})\widehat{f}(X_{n,k+1})\,\frac{h^2}{2} \\
&\quad = F(X_{n,k}) + (B^H_{t^n_{k+1}} - B^H_{t^n_k}) - \widehat{f}'(X_{n,k})\int_{t^n_k}^{t^n_{k+1}} (B^H_{t^n_{k+1}} - B^H_s)ds, \qquad \text{(A1)} \\
&\quad X_{n,0} = x_0,\ 0 \leqslant k \leqslant n-1.
\end{aligned}
$$

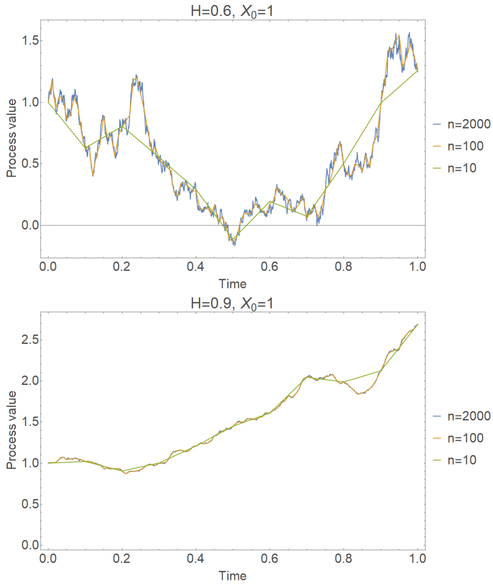


Figure 4.1: Approximation trajectories of Pearson process for conditions (4.18)

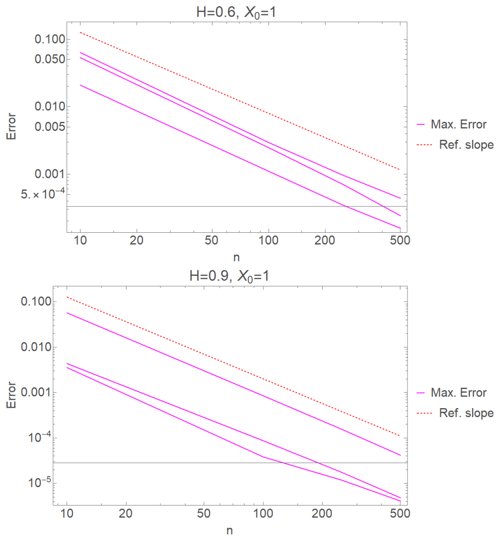


Figure 4.2: Maximum error of several approximation trajectories of Pearson process for conditions (4.18) in comparison to reference slope

To compare the theoretical and the empirical convergence rate of the Pearson process (4.18), we simulate the *exact* solution by using the approximation scheme (A1) for comparatively much smaller step size $h = 2 \cdot 10^{-3}$. We see from Fig. 4.2 that the empirical maximum error coincides with the theoretical result in Theorem 11 (reference slope).

Chapter 5

Approximation of integrated fractional Brownian motion

In this short chapter, we propose a method to calculate the integral

$$\int_{t_m^n}^{t_{m+1}^n} B_s^H \, ds. \tag{5.1}$$

Analogously to the approximation of standard Riemann integral, we suggest that the integral (5.1) can be replaced by the sum $\frac{T}{n^2}\sum_{k=1}^n B_{s_{k,m}}^H$, where $t_m^n = \frac{mT}{n}$, $0 \leqslant m \leqslant n$ and $s_{k,m}^n = t_m^n + \frac{kT}{n^2}$, $0 \leqslant k \leqslant n$. For simplicity of notation $t_m = t_m^n, s_{k,m} = s_{k,m}^n$. This approach enables simple and direct use of fractional Brownian motion simulation packages provided in most mathematical programming languages (*Wolfram Mathematica* was used for our simulations).

In order to find the convergence rate of this approximation we will need the following result (similar to Lemma 2.1 and its proof in [30]):

Lemma 4. *Let $\alpha > 1/2$ and $K \in (0, \infty)$. In addition, let Z_n, $n \in \mathbb{N}$, be a sequence of random variables such that*

$$(\mathbb{E}|Z_n|^2)^{1/2} \leqslant K \cdot n^{-\alpha}$$

for all $n \in \mathbb{N}$. Then for $\forall \epsilon : \alpha > \epsilon > 1/2$ there exists a random variable η_ε such that

$$|Z_n| \leq \eta_\varepsilon \cdot n^{-\alpha+\varepsilon} \qquad \textit{almost surely}$$

for all $n \in \mathbb{N}$. Moreover, $\mathbb{E}|\eta_\varepsilon| < \infty$.

Proof. Fix $\alpha > \epsilon > 1/2$. Then for all $\delta > 0$ from the Chebyshev–Markov inequality and the assumptions of the lemma we obtain

$$\mathbb{P}(n^{\alpha-\epsilon}|Z_n| > \delta) \leq \frac{\mathbb{E}|Z_n|^2}{\delta^2} n^{2(\alpha-\epsilon)} \leq \frac{K^2}{\delta^2} n^{-2\epsilon}.$$

Since $\alpha > \epsilon > 1/2$ we have

$$\sum_{n=1}^{\infty} \mathbb{P}(n^{\alpha-\epsilon}|Z_n| > \delta) < \infty$$

for all $\delta > 0$. The Borel–Cantelli lemma then implies that $Z_n \to 0$ almost surely for $n \to \infty$. Now set $\eta_\epsilon = \sup_n n^{\alpha-\epsilon}|Z_n|$. It follows that

$$\begin{aligned}\mathbb{E}|\eta_\epsilon|^2 &= \mathbb{E}\sup_n n^{2(\alpha-\epsilon)}|Z_n|^2 \le \mathbb{E}\sum_{n=1}^{\infty} n^{2(\alpha-\epsilon)}|Z_n|^2 \\ &= \sum_{n=1}^{\infty} n^{2(\alpha-\epsilon)}\mathbb{E}|Z_n|^2 \le K^2\sum_{n=1}^{\infty} n^{-2\epsilon} < \infty\end{aligned}$$

since $2\epsilon > 2\cdot 2^{-1} = 1$. Applying Jensen's inequality we obtain $\mathbb{E}|\eta_\epsilon| < \infty$. Indeed,

$$(\mathbb{E}|\eta_\epsilon|)^2 \le \mathbb{E}|\eta_\epsilon|^2.$$

Thus

$$\mathbb{E}|\eta_\epsilon| \le (\mathbb{E}|\eta_\epsilon|^2)^{1/2}.$$

Since $|\eta_\epsilon| < \infty$ then the assertion of the lemma now follows by

$$|Z_n| \le \left(\sup_n n^{\alpha-\epsilon}|Z_n|\right) n^{-\alpha+\varepsilon} = \eta_\varepsilon \cdot n^{-\alpha+\varepsilon}.$$

$\square$

Using this lemma we will get the rate of a. s. convergence of the difference

$$\int_{t_m}^{t_{m+1}} B_s^H ds - \frac{T}{n^2}\sum_{k=1}^{n} B_{s_{k,m}}^H.$$

In order to do this, first, we shall introduce the concept of generalized harmonic numbers (GHN). Define

$$H_n^{(r)} := \sum_{k=1}^{n} \frac{1}{k^r},$$

where $r = \sigma + it$ is a complex variable as generalized harmonic numbers.

Here are some important properties of GHN sums [44].

Proposition 6. *The following identity is true*

$$\sum_{i=1}^{n} i^{-a} H_i^{(b)} + \sum_{i=1}^{n} i^{-b} H_i^{(a)} = H_n^{(a)} H_n^{(b)} + H_n^{(a+b)}, \tag{5.2}$$

where $a, b \in \mathbb{R}$.

Proposition 7. *The following identity is true*

$$\sum_{i=1}^{n-1} H_i^{(a)} = n H_n^{(a)} - H_n^{(a-1)}, \tag{5.3}$$

where $a \in \mathbb{R}$.

Using these properties the following result can be proven.

Theorem 12. *These equalities are true:*

1. *Integral expectation*

$$\mathbb{E}\left(\int_{t_m}^{t_{m+1}} B_s^H ds\right)^2$$
$$= \frac{T^{2H+2}}{(2H+1)n^{2H+2}} \left[(m+1)^{2H+1} - m^{2H+1} - \frac{1}{2H+2}\right]$$

2. *Product sum expectation*

$$\mathbb{E} \sum_{k=1}^{n} \sum_{l=1}^{n} B_{s_{k,m}}^H B_{s_{l,m}}^H$$
$$= \frac{T^{2H}}{n^{4H}} H_n^{(-2H-1)} + \frac{T^{2H}}{n^{4H-1}} \left[H_{n(m+1)}^{(-2H)} - H_{nm}^{(-2H)} - H_n^{(-2H)}\right].$$

3. *Mixed products*

$$2\mathbb{E} \sum_{k=1}^{n} B_{s_{k,m}}^H \int_{t_m}^{t_{m+1}} B_s^H ds$$
$$= \frac{T^{2H+1}}{(2H+1)n^{2H}} \left[(m+1)^{2H+1} - m^{2H+1} + n^{-1}\right]$$
$$+ \frac{T^{2H+1}}{n^{4H+1}} \left[H_{n(m+1)}^{(-2H)} - H_{nm}^{(-2H)}\right] - \frac{2T^{2H+1}}{(2H+1)n^{4H+2}} H_n^{(-2H-1)}.$$

Proof. Using the covariance properties fBm as in [1], we obtain for integrals

$$\begin{aligned}&\mathbb{E}\left(\int_{t_m}^{t_{m+1}} B_s^H ds \int_{t_m}^{t_{m+1}} B_u^H du\right)\\&=(t_{m+1}-t_m)\int_{t_m}^{t_{m+1}} u^{2H}du-\frac{1}{2H+1}\int_0^{t_{m+1}-t_m} u^{2H+1}du.\end{aligned}$$

and for sums

$$\begin{aligned}\mathbb{E}\sum_{k=1}^{n}\sum_{l=1}^{n} B_{s_{k,m}}^H B_{s_{l,m}}^H =&\frac{T^{2H}}{2n^{4H}}\sum_{k=1}^{n}\sum_{l=1}^{n}\left[(mn+k)^{2H}+(mn+l)^{2H}-|k-l|^{2H}\right]\\=&\frac{T^{2H}}{n^{4H-1}}\left[H_{n(m+1)}^{(-2H)}-H_{nm}^{(-2H)}\right]-\frac{T^{2H}}{n^{4H}}\sum_{k=1}^{n-1}H_k^{(-2H)}.\end{aligned}$$

Now, applying Proposition 7 gives us the result.

The mixed product of integral and sum is estimated the following way

$$\begin{aligned}2\mathbb{E}\sum_{k=1}^{n} B_{s_{k,m}}^H \int_{t_m}^{t_{m+1}} B_s^H ds =& \sum_{k=1}^{n}\int_{t_m}^{t_{m+1}}\left(s_{k,m}^{2H}+s^{2H}-|s-s_{k,m}|^{2H}\right)ds\\=&\frac{T^{2H+1}}{n^{4H+1}}\left[H_{n(m+1)}^{(-2H)}-H_{nm}^{(-2H)}\right]+\frac{T^{2H+1}}{(2H+1)n^{2H}}\left[(m+1)^{2H+1}-m^{2H}\right]\\&-\frac{T^{2H+1}}{(2H+1)n^{4H+2}}\sum_{k=1}^{n}\left[(n-k)^{2H+1}+k^{2H+1}\right]\\=&\frac{T^{2H+1}}{n^{4H+1}}\left[H_{n(m+1)}^{(-2H)}-H_{nm}^{(-2H)}\right]+\frac{T^{2H+1}}{(2H+1)n^{2H}}\left[(m+1)^{2H+1}-m^{2H}\right]\\&-\frac{T^{2H+1}}{(2H+1)n^{4H+2}}\left[2H_n^{(-2H-1)}-n^{2H+1}\right].\end{aligned}$$

□

For further reasoning, we are going to need one elementary theorem from mathematical analysis.

Theorem 13. *Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a non-decreasing continuous function and let*

$$S=\sum_{i=1}^{n} f(i), \qquad I=\int_1^n f(x)dx.$$

Then

$$I + f(1) \leq S \leq I + f(n).$$

Proposition 8. *The following asymptotics occur*

$$\mathbb{E}\left(Y_{n,m,k,T}\right)^2 = O(n^{-2H-3}),$$

where

$$Y_{n,m,k,T} = \int_{t_m}^{t_{m+1}} B_s^H ds - \frac{T}{n^2}\sum_{k=1}^{n} B_{s_{k,m}}^H.$$

Proof. From Theorem 12 we get

$$\begin{aligned}
\mathbb{E}Y_{n,m,k,T}^2 =& \mathbb{E}\left(\int_{t_m}^{t_{m+1}} B_s^H ds\right)^2 + \mathbb{E}\left(\frac{T}{n^2}\sum_{k=1}^{n} B_{s_{k,m}}^H\right)^2 \\
& - 2\mathbb{E}\left(\int_{t_m}^{t_{m+1}} B_s^H ds \frac{T}{n^2}\sum_{k=1}^{n} B_{s_{k,m}}^H\right) \\
=& -\frac{T^{2H+2}}{(2H+1)(2H+2)n^{2H+2}} - \frac{T^{2H+2}}{(2H+1)n^{2H+3}} \\
& - \frac{T^{2H+2}}{n^{4H+3}} H_n^{(-2H)} + \frac{(2H+3)T^{2H+2}}{(2H+1)n^{4H+4}} H_n^{(-2H-1)} \\
\leqslant& -\frac{T^{2H+2}}{(2H+1)(2H+2)n^{2H+2}} - \frac{T^{2H+2}}{(2H+1)n^{2H+3}} \\
& - \frac{T^{2H+2}}{n^{4H+3}}\left[\frac{n^{2H+1}}{2H+1} + \frac{2H}{2H+1}\right] \\
& + \frac{(2H+3)T^{2H+2}}{(2H+1)n^{4H+4}}\left[\frac{n^{2H+2}}{2H+2} - \frac{1}{2H+2} + n^{2H+1}\right] \\
=& \frac{T^{2H+2}}{n^{2H+2}}\left[-\frac{1}{(2H+1)(2H+2)} - \frac{1}{2H+1} + \frac{2H+3}{(2H+1)(2H+2)}\right] \\
& + \frac{(2H+2)T^{2H+2}}{(2H+1)n^{2H+3}} - \frac{2HT^{2H+2}}{(2H+1)n^{4H+3}} \\
& - \frac{(2H+3)T^{2H+2}}{(2H+1)(2H+2)n^{4H+4}} \leq Cn^{-2H-3}.
\end{aligned}$$

□

Finally, from this and Lemma 4 we get the following estimate of our integrated fractional Brownian motion approximation

Lemma 5. *For* $\forall\epsilon : H + 3/2 > \epsilon > 1/2$ *there exists a random variable* $\eta_\varepsilon : \mathbb{E}|\eta_\varepsilon| < \infty$ *such that*

$$|Y_{n,m,k,T}| \leq \eta_\varepsilon \cdot n^{-H-3/2+\varepsilon} \qquad \textit{almost surely}$$

for all $n \in \mathbb{N}$.

Chapter 6

Class of the fractional stochastic differential equations with soft wall

The majority of attempts to extend classical stochastic differential equations, driven by standard Brownian motion, to those driven by fractional Brownian motion have been motivated by problems encountered in financial applications of SDEs, such as option pricing, stochastic volatility, and interest rate modeling, to name a few. However, mathematicians have not yet fully explored the potential application possibilities of stochastic processes in material sciences, especially in biology/medicine. According to Fulinski [20], certain findings in these fields *seem to open the whole new chapter in the Brownian motions story.* Upon closer inspection, natural sciences offer a unique set of requirements and challenges in the application of SDEs.

One type of challenges encountered when dealing with processes that operate in the material realm (as opposed to financial processes, for example) is the impact of the surrounding medium on the behaviour of the process itself. For instance, boundaries, which limit the propagation of process trajectories. Vojta Et al. [55] present us with a list of such boundary conditions, which they call walls and classify them dependant on process discrete expressions.

The type we are going to be concerned with is soft wall described by the recursion relation

$$x_{n+1} = x_n + \xi_n + G(x_n),$$

where x_n - process value, ξ_n - discrete fractional Gaussian noise, G - repulsive force; is far less known and demonstrates a very interesting subtle behaviour. The subtlety is produced by the introduction of repulsion instead of reflection. In contrast to the standard SDEs with reflection, where the process cannot pass the hard wall, the soft wall is repulsive but not impenetrable. As the process crosses the soft wall boundary, it experiences a force of a chosen magnitude in the opposite direction. When the process is far from the wall, the force acts weakly. Since this muffled deflection of process trajectories avoids elasticity or inelasticity assumptions it is a much more accurate mathematical model for many actual natural processes.

Therefore, we consider a one-dimensional SDE

$$Y_t = Y_0 + G(Y_t) - G(Y_0) + \int_0^t f(s, Y_s)\, ds + Z_t, \qquad t \in [0,T], \tag{6.1}$$

driven by an arbitrary stochastic process $Z = (Z_t)_{t \geqslant 0}$, $Z_0 = 0$, with continuous paths, where functions $G : \mathbb{R} \to [0,\infty)$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous. As a special case, we can take $Z = B^H$, where B^H is a fBm.

We will call such equation - *SDE with a soft wall*, where the function G is interpreted as a repulsive force, which adjusts as the process position with respect to the wall changes. For instance, we can take the exponential repulsive force profile with wall boundary at the point w, defined as

$$G(x) = G_0 \exp\{-\lambda(x-w)\},$$

characterized by amplitude G_0 and decay constant λ (see [55]).

6.1 Main results

Using function $D(x) := x - G(x)$, Equation (6.1) can be simplified to

$$D(Y_t) = D(Y_0) + \int_0^t f(s, Y_s)\, ds + Z_t. \tag{6.2}$$

6.1.1 Conditions

We proceed by introducing a set of conditions:

(**A**) *Linear growth* of f for any $t \in [0,T]$ and any $x \in \mathbb{R}$

$$|f(t,x)| \leqslant K(1+|x|).$$

(**B**) *Lipschitz continuity* of f *in space*: for any $t \in [0,T]$ and $x, y \in \mathbb{R}$

$$|f(t,x) - f(t,y)| \leqslant K|x-y|.$$

(**C**) *Hölder continuity in time*: there exists $\beta \in (0,1]$ such that for any $s, t \in [0,T]$ and any $x \in \mathbb{R}$

$$|f(s,x) - f(t,x)| \leqslant K|s-t|^{\beta}.$$

(**D**) Function D is strictly monotonic and surjective.

(**E**) There exists constant $d > 0$, such that

$$|D(x) - D(y)| \geqslant d|x-y|. \tag{6.3}$$

6.1.2 Theorems

Now, we formulate the solution existence and uniqueness theorem for the SDE (6.1).

Theorem 14. *Assume that conditions (**A**), (**B**), (**D**) and (**E**) are satisfied. Then, the Equation* (6.1) *has a unique solution* $Y \in C([0,T])$*. If stochastic process* $Z = (Z_t)_{t \geqslant 0}$*,* $Z_0 = 0$*, has a Hölder continuous paths of order* $0 < \gamma < 1$*, then* $Y \in C^{\gamma}([0,T])$*,* $\gamma \in (0,1)$*.*

Continuing, we introduce the implicit Euler scheme associated with (6.1). Let $\pi = \{t_k^n = (k/n)T, 1 \leq k \leq n\}$ be a sequence of uniform partitions of the interval $[0,T]$ and $h_n = t_k^n - t_{k-1}^n$, $1 \leq k \leq n$. We define the following implicit Euler approximation scheme:

$$D(Y_{n,k+1}) = D(Y_{n,k}) + f(t_k^n, Y_{n,k})h_n + (Z_{t_{k+1}^n} - Z_{t_k^n}), \qquad Y_{n,0} = Y_0. \tag{6.4}$$

Theorem 15. *Assume assumptions* (***A***) $-$ (***E***) *are satisfied, and* Y *is a solution of Equation* (6.1) *such that* $Y \in C^{\gamma}([0,T])$*,* $\gamma \in (0,1)$*. Then,*

$$\max_{1 \leqslant k \leqslant n} |Y_{t_k^n} - Y_{n,k}| = O_{\omega}(h_n^{\theta}),$$

where $\theta = \beta \wedge \gamma$*.*

Corollary 1. *Under conditions of Theorem 15, it follows that*

$$\sup_{0\leq t\leq T} |Y_t - Y_t^n| = O_\omega(h_n^\theta), \quad \gamma \in (0,1), \tag{6.5}$$

where

$$Y_t^n = Y_{n,k} + \frac{t-t_k^n}{h_n}(Y_{n,k+1} - Y_{n,k}), \quad t \in (t_k^n, t_{k+1}^n],\ k = 0,\ldots,n-1.$$

We now proceed by application of the results obtained for Equation (8.27) to equations with additional restrictions on the diffusion coefficient. Let us consider SDE

$$X_t = X_0 + \Phi(X_t) - \Phi(X_0) + \int_0^t \alpha(X_s)\,ds + \int_0^t \sigma(X_s)\,dZ_s, \qquad t \in [0,T], \tag{6.6}$$

where Φ, α, σ are continuous functions, Z is a process with Hölder continuous paths of order $1/2 < \gamma < 1$ and strictly positive diffusion coefficient, i.e., $\inf_{x\in\mathbb{R}} \sigma(x) > 0$. The stochastic integral in Eq. (6.6) is a pathwise Riemann–Stieltjes integral, and thus, the whole equation is understood as a pathwise Riemann–Stieltjes integral equation. A pathwise Riemann–Stieltjes integral is well defined if $\sigma(X)$ is Hölder continuous with order λ and such that $\lambda + \gamma > 1$.

For the following theorem, we need condition:

(H) $\Phi : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, and there exists a constant $0 \leqslant c < 1$ such that $\Phi'(x) \leqslant c$ for all $x \in \mathbb{R}$.

Theorem 16. *Assume that $\alpha(x)$ and $\sigma(x)$ are continuously differentiable functions and $\inf_{x\in\mathbb{R}} \sigma(x) > 0$. If the function $\Phi(x)$ satisfies condition **(H)** and for some constant $C > 0$ and any $x \in \mathbb{R}$,*

$$|\alpha'(x)| \leqslant C, \qquad |\sigma'(x)| \leqslant C, \qquad \left|\frac{\alpha(x)}{\sigma(x)}\right| \leqslant C \tag{6.7}$$

then SDE (6.6) *has a unique solution $X \in C^\gamma([0,T])$, $1/2 < \gamma < 1$.*

Theorem 17. *Assume that conditions of Theorem 16 hold. Then,*

$$\max_{1\leqslant k\leqslant n} |X_{t_k^n} - X_{n,k}| = O_\omega(h_n^\gamma), \qquad \gamma \in (1/2, 1),$$

where $X_{n,k} = F^{-1}(Y_{n,k})$ and F^{-1} is the inverse Lamperti transform.

6.2 Remarks on the condition (E)

At first glance condition **(E)** can seem unusual and restrictive (especially in conjunction with condition **(D)**). However, a wide class of functions satisfies it. We will propose only several more interesting examples of such functions.

Remark 5. *Assume that the function $D:\mathbb{R}\to\mathbb{R}$ is bi-Lipschitz (i.e. D is Lipschitz and its inverse function D^{-1} is also Lipschitz). Then inequality* (6.3) *is satisfied.*

For any $\widehat{x},\widehat{y}\in\mathbb{R}$ there exsist $x,y\in\mathbb{R}$ such that $D^{-1}(\widehat{x})=x$ and $D^{-1}(\widehat{y})=y$. Since D^{-1} is Lipschitz function then for some L

$$|x-y|=|D^{-1}(\widehat{x})-D^{-1}(\widehat{y})|\leqslant L|\widehat{x}-\widehat{y}|=L|D(x)-D(y)|. \tag{6.8}$$

Remark 6. *Assume that the function $D:\mathbb{R}\to\mathbb{R}$ is differentiable, invertable and such that $|D'|_\infty\geqslant c>0$, then inequality* (6.3) *is satisfied.*

Since $|D'|_\infty\geqslant c>0$ and by using Inverse Function Theorem we get

$$|(D^{-1})'(x)|=\left|\frac{1}{D'(D^{-1}(x))}\right|\leqslant\frac{1}{c}.$$

Thus, the function $D^{-1}(x)$ is Lipschitz, and inequality (6.3) *is satisfied.*

Remark 7. *If $G:\mathbb{R}\to\mathbb{R}$ is continuously differentiable function and there exists a constant $0\leqslant c<1$ such that $G'(x)\leqslant c$, for all $x\in\mathbb{R}$, then $D(x)$ is strictly increasing and $|D(x)-D(y)|\geqslant(1-c)|x-y|$. Indeed, since $D'(x)\geqslant 1-c$ for all $x\in\mathbb{R}$ then $D(x)$ is strictly increasing and from Lagrange's mean value theorem it follows that*

$$(1-c)|x-y|\leqslant|(D(x)-D(y)|.$$

Remark 8. *If $G(x)$ is Lipschitz continuous with Lipschitz constant $0\leqslant c<1$, then $|D(x)-D(y)|\geqslant(1-c)|x-y|$. Since $|G(x)-G(y)|\leqslant c|x-y|$ we see at once that*

$$|D(x)-D(y)|\geqslant|x-y|-|G(x)-G(y)|\geqslant(1-c)|x-y|.$$

6.3 Proofs of theorems

To start with, we will construct and prove one auxiliary result. Consider the deterministic differential equation on $\mathbb{R}$

$$x_t = x_0 + G(x_t) - G(x_0) + \int_0^t f(s, x_s)\,ds + \varphi_t, \qquad t \in [0,T], \tag{6.9}$$

where $x_0 \in \mathbb{R}$, $\varphi \in C([0,T])$, $T > 0$, $\varphi(0) = 0$, functions $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ are continuous.

Theorem 18. *Assume that conditions (**A**), (**B**), (**D**) and (**E**) are satisfied. Then, the Equation* (6.9) *has a unique solution* $x \in C([0,T])$. *If* $\varphi \in C^\gamma([0,T])$, $\gamma \in (0,1)$, *then* $x \in C^\gamma([0,T])$, $\gamma \in (0,1)$.

Proof. Recall that the function $D(x) = x - G(x)$ is strictly monotonic and continuous. Then, it has the inverse $D^{-1}(x)$, which is strictly monotonic and continuous too.

Existence. We will apply the Picard iteration method. Define the Picard iterations sequence as follows. Let

$$D(x_t^{m+1}) = D(x_0) + \int_0^t f(s, x_s^m)\,ds + \varphi_t, \qquad x_t^0 = x_0,\ x_0^m = x_0,\ m \geqslant 0.$$

Let $y_t^{m+1} := D(x_t^{m+1})$. Then

$$y_t^{m+1} = D(x_0) + \int_0^t f(s, x_s^m)\,ds + \varphi_t$$

is a continuous function. Since $y_t^1 = D(x_0) + \int_0^t f(s, x_0)\,ds + \varphi_t$, it is evident that y^1 is a continuous function. By this and continuity of the inverse D^{-1} one obtains that $x_t^1 = D^{-1}(y_t^1)$ is continuous too. In the general case, if x_t^m is a continuous function, then y_t^{m+1} is a continuous function too. Thus, $x_t^{m+1} = D^{-1}(y_t^{m+1})$ is continuous.

We now turn to the proof that the sequence $(x_t^m)_{0 \leqslant t \leqslant T}$ converges uniformly on $[0,T]$ to a continuous limiting function $\tilde{x} \in C([0,T])$.

Note that

$$D(x_t^{m+1}) - D(x_t^m) = \int_0^t [f(s, x_s^m) - f(s, x_s^{m-1})]\,ds.$$

Thus, from the conditions (**B**) and (**D**), we obtain

$$d|x_t^{m+1} - x_t^m| \leqslant |(D(x_t^{m+1}) - D(x_t^m)|$$

and

$$|x_t^{m+1} - x_t^m| \leqslant \frac{K}{d} \int_0^t |x_s^m - x_s^{m-1}|\, ds.$$

By this inequality, we have

$$\begin{aligned}|x_t^{m+1} - x_t^m| \leqslant & \frac{K^2}{d^2} \int_0^t \int_0^{s_1} |x_{s_2}^{m-1} - x_{s_2}^{m-2}|\, ds_2\ ds_1 \\ \leqslant & \frac{K^m}{d^m} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{m-1}} |x_{s_m}^1 - x_{s_m}^0|\, ds_m \ldots ds_2 ds_1.\end{aligned}$$

From inequality

$$|x_t^1 - x_0| \leqslant \frac{1}{d}\Big(\int_0^t |f(s,x_0)|\, ds + |\varphi_t|\Big),$$

it follows that

$$\sup_{0\leqslant t\leqslant T} |x_t^1 - x_0| \leqslant C_T, \qquad C_T := d^{-1}\Big(T \sup_{0\leqslant t\leqslant T} |f(t,x_0)| + \sup_{0\leqslant t\leqslant T} |\varphi_t|\Big).$$

Consequently,

$$|x_t^{m+1} - x_t^m| \leqslant \frac{(Kt)^m}{d^m m!} C_T.$$

Thus,

$$\sum_{m=0}^{\infty} \sup_{0\leqslant t\leqslant T} |x_t^{m+1} - x_t^m| < \infty$$

and the sequence $x^m = (x_t^m)_{0\leqslant t\leqslant T}$ converge uniformly on $[0,T]$ to a continuous limiting function $\tilde{x} \in C([0,T])$.

D is a continuous function, and for all $t \in [0,T]$,

$$\begin{aligned}&\Big| D(\tilde{x}_t) - D(x_0) - \int_0^t f(s,\tilde{x}_s)\, ds - \varphi_t \Big| \\ &\quad \leqslant |D(x_t^m) - D(\tilde{x}_t)| + \int_0^t |f(s,x_s^{m-1}) - f(s,\tilde{x}_s)|\, ds \\ &\quad \leqslant \sup_{0\leqslant t\leqslant T} |D(x_t^m) - D(\tilde{x}_t)| + TK \sup_{0\leqslant t\leqslant T} |\tilde{x}_t - x_t^{m-1}| \xrightarrow[m\to\infty]{} 0;\end{aligned}$$

hence, $\tilde{x}$ is a solution of Equation (6.9).

Next, we prove that $\tilde{x} \in C^\gamma([0,T])$, $\gamma \in (0,1)$, if $\varphi \in C^\gamma([0,T])$,

$\gamma \in (0,1)$. By the similar method as above, it follows that

$$|\tilde{x}_t - \tilde{x}_s| \leqslant \frac{1}{d}\int_s^t |f(u,\tilde{x}_u)|\,du + d^{-1}|\varphi_t - \varphi_s|.$$

Finally, from condition (**A**), we obtain that

$$\begin{aligned}|\tilde{x}_t - \tilde{x}_s| \leqslant & Kd^{-1}\int_s^t (1+|\tilde{x}_u|)\,du + d^{-1}|\varphi|_\gamma (t-s)^\gamma \\ \leqslant & Kd^{-1}(1+|\tilde{x}|_\infty)(t-s) + d^{-1}|\varphi|_\gamma (t-s)^\gamma \\ \leqslant & d^{-1}(K(1+|\tilde{x}|_\infty)(T\vee 1) + |\varphi|_\gamma)(t-s)^\gamma \end{aligned} \tag{6.10}$$

and the proof is complete.

Uniqueness. Assume that y is another solution of (6.9). The uniqueness of the solution $\tilde{x}$ of (6.9) follows from the inequality

$$\sup_{0\leqslant t\leqslant T} |\tilde{x}_t - y_t| \leqslant Kd^{-1}\int_0^T \sup_{0\leqslant s\leqslant t} |\tilde{x}_s - y_s|\,ds$$

and Gronwall's lemma. □

6.3.1 Proof of Theorems 14 and 15

Proof of Theorem 14. The existence and uniqueness of a solution of Equation (6.1) is derived directly from the deterministic version of Theorem 18. □

Proof of Theorem 15. We rewrite the approximation (6.4) in the following way

$$\begin{aligned} D(Y_{n,k+1}) = & D(Y_{n,k}) + f(t_k^n, Y_{n,k})h_n + (Z_{t_{k+1}^n} - Z_{t_k^n}) \\ = & D(Y_0) + h_n \sum_{j=0}^{k} f(t_j^n, Y_{n,j}) + Z_{t_{k+1}^n} \end{aligned} \tag{6.11}$$

and discretize the process Y by

$$
\begin{aligned}
D(Y_{t^n_{k+1}}) =& D(Y_{t^n_k}) + f(t^n_k, Y_{t^n_k})h_n + (Z_{t^n_{k+1}} - Z_{t^n_k}) \\
&+ \int_{t^n_k}^{t^n_{k+1}} (f(s,Y_s) - f(t^n_k, Y_{t^n_k}))ds \\
=& D(Y_0) + h_n \sum_{j=0}^{k} f(t^n_j, Y_{t^n_j}) + Z_{t^n_{k+1}} + R_{n,k+1}, \qquad (6.12)
\end{aligned}
$$

where

$$
R_{n,k+1} := \sum_{j=0}^{k} \int_{t^n_j}^{t^n_{j+1}} (f(s,Y_s) - f(t^n_j, Y_{t^n_j}))ds.
$$

Then,

$$
D(Y_{t^n_{k+1}}) - D(Y_{n,k+1}) = h_n \sum_{j=1}^{k} [f(t^n_j, Y_{t^n_j}) - f(t^n_j, Y_{n,j})] + R_{n,k+1}.
$$

Similarly, as in proof of Theorem 18, we obtain

$$
|Y_{t^n_{k+1}} - Y_{n,k+1}| \leqslant \frac{Kh_n}{d} \sum_{j=1}^{k} |Y_{t^n_j} - Y_{n,j}| + d^{-1}|R_{n,k+1}|. \qquad (6.13)
$$

It remains to estimate $|R_{n,k+1}|$. Assume that $t \in [t^n_k, t^n_{k+1})$. Then (see (6.10)),

$$
|Y_t - Y_{t^n_k}| \leqslant d^{-1}[K(1+|Y|_\infty) + |Z|_\gamma]h_n^\gamma.
$$

Thus,

$$
\begin{aligned}
|R_{n,k+1}| \leqslant& n \max_{0\leqslant k\leqslant n-1} \int_{t^n_k}^{t^n_{k+1}} |f(s,Y_s) - f(t^n_k, Y_{t^n_k})|\, ds \\
\leqslant& Kn \max_{0\leqslant k\leqslant n-1} \int_{t^n_k}^{t^n_{k+1}} (s - t^n_k)^\beta ds + Kn \max_{0\leqslant k\leqslant n-1} \int_{t^n_k}^{t^n_{k+1}} |Y_s - Y_{t^n_k}|\, ds \\
\leqslant& \frac{KT}{1+\beta} h_n^\beta + \frac{KT}{d}[K(1+|Y|_\infty) + |Z|_\gamma]h_n^\gamma \leqslant C_\omega h_n^\theta, \qquad (6.14)
\end{aligned}
$$

where $\theta = \beta \wedge \gamma$,

$$
C_\omega := \frac{KT}{1+\beta} h_n^{\beta-\theta} + \frac{KT}{d}[K(1+|Y|_\infty) + |Z|_\gamma]h_n^{\gamma-\theta}.
$$

Combining inequalities (6.13), (6.14) and the discrete version of Gronwall's Lemma 6, we obtain

$$\max_{0\leqslant k\leqslant n-1} |Y_{t^n_{k+1}} - Y_{n,k+1}| \leqslant C_\omega e^{KTd^{-1}} h_n^\theta .$$

$\square$

Proof of Corollary 1. By definition of Y^n, for any $t \in (t^n_k, t^n_{k+1}]$,

$$\begin{aligned} Y_t - Y_t^n &= Y_t - \frac{t-t^n_k}{h_n} Y_{n,k+1} - \frac{t^n_{k+1}-t}{h_n} Y_{n,k} \\ &= \frac{t^n_k - t}{h_n}\left[\int_t^{t^n_{k+1}} f(s,Y_s)\,ds + Z_{t^n_{k+1}} - Z_t\right] \\ &\quad + \frac{t^n_{k+1}-t}{h_n}\left[\int_{t^n_k}^{t} f(s,X_s)\,ds + Z_t - Z_{t^n_k}\right] \\ &\quad + \frac{t-t^n_k}{h_n}(Y_{t^n_{k+1}} - Y_{n,k+1}) + \frac{t^n_{k+1}-t}{h_n}(Y_{t^n_k} - Y_{n,k}). \end{aligned}$$

We see at once that the asymptotic behavior of the first two terms is $O_\omega(n^{-\gamma})$, which is clear from inequality (6.10), and the estimation of the last two terms follows from Theorem 15. This completes the proof. $\square$

6.3.2 Proof of Theorems 16 and 17

If coefficients $\alpha(x)$ and $\sigma(x)$ of Equation (6.6) are Lipschitz functions and $X \in C^\gamma(0,T)$, $1/2<\gamma<1$, then Equation (6.6) is well defined. We need to find a process $X \in C^\gamma(0,T)$, $1/2<\gamma<1$ satisfying Equation (6.6).

Consider SDE

$$Y_t = Y_0 + V(Y_t) - V(Y_0) + \int_0^t f(Y_s)\,ds + Z_t, \tag{6.15}$$

where

$$f(x) = \frac{\alpha(F^{-1}(x))}{\sigma(F^{-1}(x))}, \qquad V(x) = \int_0^x g(u)\,du,$$

$$g(u) = \Phi'(F^{-1}(u)), \qquad F(x) = \int_0^x \frac{1}{\sigma(y)}\,dy.$$

Recall that the Lamperti transform $F(x)$ has the inverse function $F^{-1}:\mathbb{R}\to\mathbb{R}$, which is strictly increasing and differentiable.

The proof of Theorem 16 consists of two steps. First, we find the conditions under which the Equation (6.15) has a unique solution in $C^{\gamma}(0,T): 1/2 < \gamma < 1$.

Secondly, we prove that $X_t = F^{-1}(Y_t), Y_0 = F(X_0)$ satisfies Equation (6.6).

Proof of Theorem 16. We find requirements for coefficients $\alpha(x)$ and $\sigma(x)$ in (6.15) to satisfy condition (**B**).

Denote

$$f(x) = \widehat{f}(F^{-1}(x)), \qquad \widehat{f}(x) = \frac{\alpha(x)}{\sigma(x)}.$$

Since the inverse function $F^{-1}: \mathbb{R} \to \mathbb{R}$ is differentiable, then

$$(F^{-1})'(x) = \sigma(F^{-1}(x)), \qquad x \in \mathbb{R}$$

and

$$\begin{aligned} f'(x) =& (\widehat{f}(F^{-1}(x)))' = \widehat{f}'(F^{-1}(x))(F^{-1})'(x) \\ =& \widehat{f}'(F^{-1}(x))\sigma(F^{-1}(x)) = \Big(\alpha' - \frac{\alpha\sigma'}{\sigma}\Big)(F^{-1}(x)). \end{aligned}$$

Conditions of the theorem imply that

$$|f'(x)| = \left|\Big(\alpha' - \frac{\alpha\sigma'}{\sigma}\Big)(F^{-1}(x))\right| \leqslant C + C^2.$$

Thus, conditions (**A**) and (**B**) are satisfied at the same time. Finally, condition (**H**) is satisfied for the function $V(x)$. Indeed,
since $V'(x) = \Phi'(F^{-1}(x))$ and $\Phi'(x) \leqslant c$, then $V'(x) \leqslant c$. From Remark 7, we obtain that conditions (**D**)–(**E**) are satisfied.

Thus, $Y \in C^{\gamma}(0,T): 1/2 < \gamma < 1$ and is a unique solution of Equation (6.15).

Now, we return to the consideration of the process $X_t = F^{-1}(Y_t)$. Function $\sigma(F^{-1}(x))$ is continuously differentiable. Indeed,

$$(\sigma(F^{-1}(x)))' = \sigma'(F^{-1}(x))\sigma(F^{-1}(x))$$

and the right side of the equality is a composition of two continuous functions $\sigma'(x)\sigma(x)$ and $F^{-1}(x)$. Thus, function $(F^{-1})'(x) = \sigma(F^{-1}(x))$ is locally Lipschitz, and the process $\sigma(F^{-1}(Y_t))$ is bounded on $[0,T]$, i.e., there exists a random variable $M(\omega)$ such that

$$\sup_{0\leqslant t\leqslant T} |\sigma(F^{-1}(Y_t))| \leqslant M(\omega).$$

Consequently,

$$|X_t - X_s| = |F^{-1}(Y_t) - F^{-1}(Y_s)| \leqslant M|Y_t - Y_s| \leqslant MG_{\gamma,T}|t-s|^{\gamma},$$

where $G_{\gamma,T}$ is a random variable. Therefore, we obtain $X \in C^{\gamma}([0,T])$, $1/2 < \gamma < 1$.

Now, we prove that the process $X_t = F^{-1}(Y_t)$, $Y_0 = F(X_0)$ satisfies Equation (6.6). Note that the function $F'(x)$ is locally Lipschitz. Indeed,

$$F''(x) = -\frac{\sigma'(x)}{\sigma^2(x)}, \qquad \inf_{x\in\mathbb{R}} \sigma(x) > 0,$$

and $\sigma(x)$ is a continuous function, then $F''(x)$ is a continuous function, and therefore, $F'(x)$ is locally Lipschitz. In addition, we will note that functions $V(x)$ and $\Phi(x)$ are Lipschitz.

For $X, Y \in C^{\gamma}([0,T])$, $1/2 < \gamma < 1$, the processes $(F^{-1})'(Y_t)$, $V'(Y_t)$, $F'(X_t)$, and $\Phi'(X_t)$ are Riemann–Stieltjes integrable with respect to X and Y (see Theorem 6). Now, we can apply the chain rule formula for $F^{-1}(X_t)$, $V(Y_t)$ and $\Phi(X_t)$ (see Theorem 6). By the chain rule and substitution rule (see Proposition 3), we obtain

$$\begin{aligned}
V(Y_t) - V(Y_0) &= \int_0^t V'(Y_s)\,dY_s = \int_0^t g(Y_s)\,dY_s,\\
\Phi(X_t) - \Phi(X_0) &= \int_0^t \Phi'(X_s)\,dX_s,
\end{aligned}$$

and

$$
\begin{aligned}
X_t =& X_0+\int_0^t (F^{-1})'(Y_s)\,dY_s=(F^{-1})(Y_0)+\int_0^t \sigma(F^{-1}(Y_s))\,dY_s\\
=& X_0+\int_0^t \sigma(X_s)\,dV(Y_s)+\int_0^t \sigma(X_s)f(F(X_s))\,ds+\int_0^t \sigma(X_s)\,dZ_s\\
=& X_0+\int_0^t \sigma(X_s)g(F(X_s))\,dF(X_s)+\int_0^t \alpha(X_s)\,ds+\int_0^t \sigma(X_s)\,dZ_s\\
=& X_0+\int_0^t \sigma(X_s)g(F(X_s))\sigma^{-1}(X_s)\,dX_s+\int_0^t \alpha(X_s)\,ds+\int_0^t \sigma(X_s)\,dZ_s\\
=& X_0+\int_0^t g(F(X_s))\,dX_s+\int_0^t \alpha(X_s)\,ds+\int_0^t \sigma(X_s)\,dZ_s\\
=& X_0+\int_0^t \Phi'(X_s)\,dX_s+\int_0^t \alpha(X_s)\,ds+\int_0^t \sigma(X_s)\,dZ_s\\
=& X_0+\Phi(X_t)-\Phi(x_0)+\int_0^t \alpha(X_s)\,ds+\int_0^t \sigma(X_s)\,dZ_s.
\end{aligned}
$$

Thus, X is a solution of the Equation (6.6), and this solution is unique (it follows from the uniqueness of the solution Y and properties of the Lamperti transform F). □

Proof of Theorem 17. First, observe that

$$
\begin{aligned}
|X_{t_k^n}-F^{-1}(Y_{n,k})| =& |F^{-1}(Y_{t_k^n})-F^{-1}(Y_{n,k})|\\
=& |\sigma(F^{-1}(Y_{t_k^n}+\theta_{n,k}(Y_{t_k^n}-Y_{n,k})))|\cdot|Y_{t_k^n}-Y_{n,k}|,
\end{aligned}
$$

where $\theta_{n,k}\in(0,1)$. Theorem 15 implies that

$$
\max_{1\leqslant k\leqslant n}|Y_{t_k^n}+\theta_{n,k}(Y_{t_k^n}-Y_{n,k})|\leqslant \sup_{0\leqslant t\leqslant T}|Y_t|+C_\omega h_n^\gamma,
$$

where C_ω is a random variable. Since the function $\sigma(F^{-1}(x))$ is locally Lipschitz, we have

$$
\sigma(F^{-1}(Y_{t_k^n}+\theta_{n,k}(Y_{t_k^n}-Y_{n,k})))
$$

is bounded for almost all ω. Applying Theorem 15, we conclude that

$$
\max_{1\leqslant k\leqslant n}|X_{t_k^n}-F^{-1}(Y_{n,k})|=O_\omega(h_n^\gamma),\qquad \gamma\in(1/2,1).
$$

□

6.4 Further investigations

6.4.1 Fractional Pearson Diffusion Process with Soft Wall

First, as an example of a stochastic process, which satisfies conditions (**A**)–(**E**), we will consider the fractional version of the Pearson diffusion process.

Let

$$X_t = x_0 + \Phi(X_t) - \Phi(x_0) + \int_0^t \alpha(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s^H, \qquad t \geq 0, \tag{6.16}$$

with

$$\alpha(x) = b - ax, \qquad \sigma(x) = \sqrt{\sigma_0 + \sigma_1 x + \sigma_2 x^2}, \qquad \sigma_2 > 0.$$

Assume that σ_i, $i = 0,1,2$ are such that the square root is well defined and $\inf_{x\in\mathbb{R}} \sigma(x) > 0$, i.e., $\sigma_0 + \sigma_1 x + \sigma_2 x^2 > 0$.

Note that

$$|\alpha'(x)| \leqslant |a|, \qquad \sigma'(x) = \frac{\sigma_1 + 2\sigma_2 x}{2\sqrt{\sigma_0 + \sigma_1 x + \sigma_2 x^2}}.$$

Since $\sigma_0 + \sigma_1 x + \sigma_2 x^2 > 0$, when $\sigma_1^2 - 4\sigma_2\sigma_0 < 0$ for $\sigma_2 > 0$, we get

$$\sigma_0 + \sigma_1 x + \sigma_2 x^2 \geqslant \sigma_2 \Big(x + \frac{\sigma_1}{2\sigma_2}\Big)^2,$$

then

$$|\sigma'(x)| \leqslant \frac{|\sigma_1 + 2\sigma_2 x|}{2\sqrt{\sigma_2}|x + \frac{\sigma_1}{2\sigma_2}|} = \sqrt{\sigma_2}.$$

It remains to prove the existence of a constant $K > 0 : \frac{|\alpha(x)|}{|\sigma(x)|} \leqslant K$. By function

$$\frac{b - ax}{\sqrt{\sigma_0 + \sigma_1 x + \sigma_2 x^2}}$$

being continuous and therefore bounded on a compact set, we obtain

$$\left|\frac{\alpha(x)}{\sigma(x)}\right| \leqslant K$$

for some constant $K > 0$.

Applying the Lamperti transform F to Equation (6.16), we obtain

$$Y_t = y_0 + V(Y_t) - V(y_0) + \int_0^t \frac{b - aY_s}{\sqrt{\sigma_0 + \sigma_1 Y_s + \sigma_2 Y_s^2}}\, ds + B_t^H,$$

where $Y_t = F(X_t), y_0 = F(x_0)$ and

$$V(x) = \int_0^x g(u)\, du, \qquad g(u) = \Phi'(F^{-1}(u)).$$

Indeed,

$$\begin{aligned} Y_t = F(X_t) &= F(x_0) + \int_0^t F'(X_s)\, dX_s \\ &= F(x_0) + \int_0^t \frac{1}{\sigma(X_s)}\, d\Phi(X_s) + \int_0^t \frac{\alpha(X_s)}{\sigma(X_s)}\, ds + B_t^H. \end{aligned}$$

Since $(F^{-1})'(x) = \sigma(F^{-1}(x))$ then

$$\begin{aligned} \int_0^t \frac{1}{\sigma(X_s)}\, d\Phi(X_s) &= \int_0^t \frac{\Phi'(X_s)}{\sigma(X_s)}\, dX_s \\ &= \int_0^t \frac{\Phi'(F^{-1}(Y_s))}{\sigma(F^{-1}(Y_s))}\, dF^{-1}(Y_s) = \int_0^t \Phi'(F^{-1}(Y_s))\, dY_s. \end{aligned}$$

The approximation scheme for the process (6.16) has a form

$$X_{n,k} = F^{-1}(Y_{n,k}),$$

where

$$\begin{aligned} &Y_{n,k+1} - V(Y_{n,k+1}) \\ &\qquad = Y_{n,k} - V(Y_{n,k}) + \frac{b - aY_{n,k}}{\sqrt{\sigma_0 + \sigma_1 Y_{n,k} + \sigma_2 Y_{n,k}^2}}\, h_n + (B_{t_{k+1}^n}^H - B_{t_k^n}^H). \end{aligned}$$

6.4.2 Modelling. Fractional Vasicek Process

Let

$$X_t = x_0 + \int_0^t (\beta - \alpha X(s))\, ds + \sigma B^H(t), \quad t \geq 0, \quad \alpha, \beta \in \mathbb{R},\ \sigma \geq 0. \quad (6.17)$$

Profile of Soft-Wall Resistant Force

We introduce a soft-wall resistant force G into the process (6.17) and obtain

$$X_t = x_0 + G(X_t) - G(x_0) + \int_0^t (\beta - \alpha X(s))\, ds + \sigma B^H(t), \qquad (6.18)$$

where $t \geq 0$, $\alpha, \beta \in \mathbb{R}$, $\sigma \geq 0$.

Force G has the following conditions (**D**)–(**E**), satisfying profile

$$G(x) = \begin{cases} G_1(x) = k_1 e^{-\lambda(x-a_1)}, & \text{if } x > a_1 \\ G_2(x) = k_2(a_1 - x) + G_1(a_1), & \text{if } x \leq a_1, \end{cases} \qquad (6.19)$$

where $\lambda \in (0,1)$; $a_1, k_1, k_2 > 0$.

This specific construction of force G is chosen in order to simulate the rapid change of properties in a process surrounding medium by the simplest controllable means—linear function.

As we can see from Figure 6.1a, profile (6.19) produces a rapid increase of resistance after crossing a soft-wall boundary at a_1. However, conditions for the function G and its derivative G' do not limit the number of these changes. For example, we can have a two-step change in the force profile. Naturally, these two steps can both increase the force change rate (Figure 6.1b), or the first step can increase the rate and the second one decrease it (Figure 6.1c).

Adding another inflection point a_2 to profile (6.19), we obtain the following two-step profile

$$G(x) = \begin{cases} G_1(x) = k_1 e^{-\lambda(x-a_1)}, & \text{if } x > a_1 \\ G_2(x) = k_2(a_1 - x) + G_1(a_1), & \text{if } a_2 \leq x \leq a_1 \\ k_3(a_2 - x) + G_2(a_2), & \text{if } x < a_2, \end{cases} \qquad (6.20)$$

where $\lambda \in (0,1)$; $a_1, a_2, k_1, k_2, k_3 > 0$.

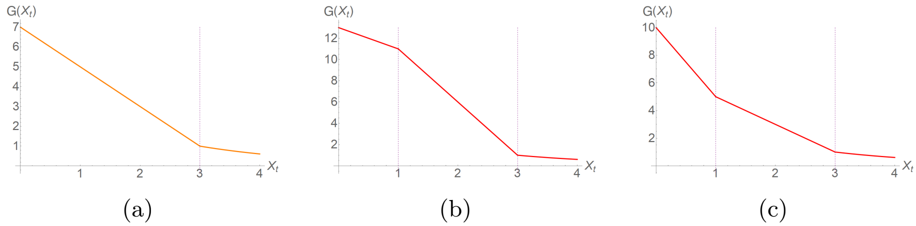


Figure 6.1: Force G profiles. (**a**) Single soft-wall. (**b**) Double "increase–decrease" soft-wall. (**c**) Double "increase–increase" soft-wall.

Process Trajectories Simulation under Soft-wall conditions

Using process approximation scheme (6.4) for the fractional Vasicek process (6.18) and soft-wall resistant force profiles (6.19) and (6.20) illustrated bellow in Figure 6.2

Figure 6.2: Single (6.19) and double "increase–increase" (6.20) force G profiles for $k_1 = 0.001, k_2 = 2,\ k_3 = 20, a_1 = 2.3, a_2 = 1.7$.

we obtain the following trajectories of the fractional Vasicek process.

As we can see from Figure 6.3 by comparing soft-wall process trajectories with trajectories without soft-wall resistance force, force G "pushes" the process trajectories towards the boundary. The strength of this "push" is dependent on the magnitude of derivative G' (large enough derivatives have almost a straightening effect on the trajectories).

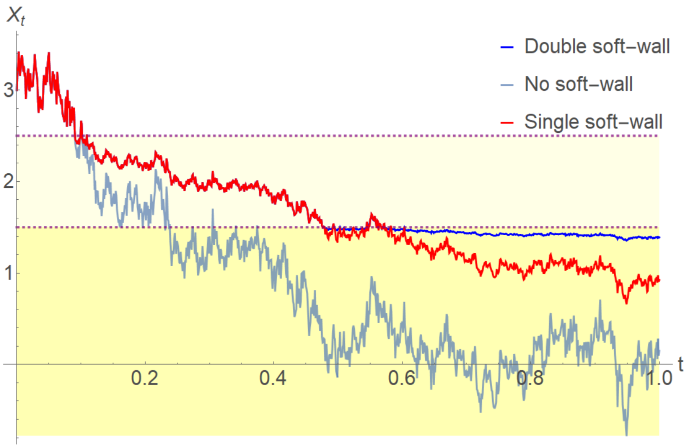


Figure 6.3: Trajectories of the fractional Vasicek process for $\alpha = 1, \beta = 3, H = 0.3, \lambda = 0.5, T = 1, n = 1000$.

Even more fascinating soft-wall behaviour can be observed for a double "increase–decrease" soft-wall force profile.

In Figure 6.4 we clearly see that the volatility of trajectories is not dependent on the value of G, but on the magnitude of its derivative G'. Hence, the double soft-wall force profile (Figure 6.5) simulates some kind of resistant membrane between boundaries a_1 and a_2 being punctured by the fractional Vasicek process (Figure 6.4). Notice how, after leaving the area $[a_2, a_1]$, the increments of the soft-wall affected trajectories become almost identical to the increments of the standard process.

Figure 6.4: Trajectories of the fractional Vasicek process for $\alpha = 1, \beta = 5, H = 0.3, \lambda = 0.5, T = 1, n = 1000$.

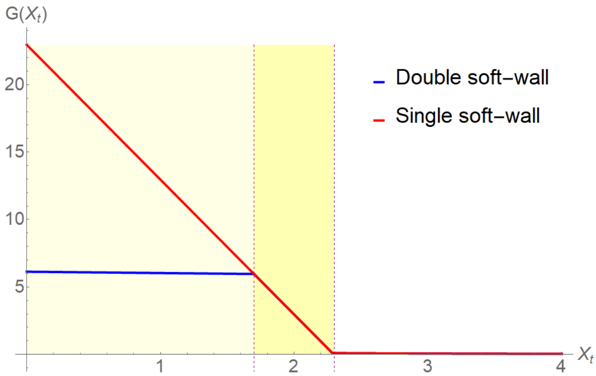


Figure 6.5: Single (6.19) and double “increase–decrease” (6.20) force G profiles for $k_1 = 0.001, k_2 = 10, k_3 = 0.1, a_1 = 2.3, a_2 = 1.7$.

6.4.3 Hurst index estimates

Similarly to Theorem 9 we construct a strongly consistent and asymptotically normal estimator of the Hurst parameter H from discrete observations of a single sample path.

Assume that in (6.1), function $G(x)$ (subsequently and function $D(x)$) is known. Then

Theorem 19. *Let Y be a solution of equation (6.1), where $Z = B^H$ is a fBm with $0 < H < 1$. Define the Hurst index estimator*

$$\widehat{H}_n = \frac{1}{2} - \frac{1}{2\ln 2} \ln \left(\frac{\sum_{k=1}^{2n-1} \left(\Delta^{(2)}_{2n,k} D(Y) \right)^2}{\sum_{k=1}^{n-1} \left(\Delta^{(2)}_{n,k} D(Y) \right)^2} \right).$$

Then

$$\widehat{H}_n \xrightarrow{a.s.} H, \qquad \text{if} \quad 0 < \beta \leqslant 1$$

$$\widehat{H}_n = H + O_\omega\left(\left(\frac{\ln n}{n} \right)^{1/2} \right), \qquad \text{if} \quad 0 \vee (H - 1/4) < \beta \leqslant 1,$$

$$2\ln 2 \sqrt{n} (\widehat{H}_n - H) \xrightarrow{d} \mathcal{N}(0, \sigma_H^2), \qquad \text{if} \quad 0 < \beta \leqslant 1,$$

with a known variance σ_H^2 defined below in Appendix A.

Proof. Fix $0 < \gamma < H$, sufficiently close to H, and such that $\gamma = H - \varepsilon$, where $0 < \varepsilon < 1/4$. Denote $\theta = \beta \wedge \gamma$. Due to $Y \in C^\gamma([0,T])$, it follows that (see [38], p. 78)

$$\begin{aligned}|f(s,Y_s)-f(t,Y_t)| &\leqslant K|s-t|^{\beta}+K|Y_s-Y_t| \\ &\leqslant K(T^{\beta-\theta}+|Y|_{\gamma}T^{\gamma-\theta})|s-t|^{\theta},\end{aligned} \tag{6.21}$$

for $s,t\in[0,T]$, where the constant K is defined in (**B**) and (**C**). From (6.21) it follows that

$$\begin{aligned}\Delta^{(2)}_{n,k}D(Y) &= \left(\int_{t_k^n}^{t_{k+1}^n} f(t,Y_t)\,dt-\int_{t_{k-1}^n}^{t_k^n} f(t,Y_t)\,dt\right)+\Delta^{(2)}_{n,k}B^H \\ &= O_{\omega}(h_n^{1+\theta})+\Delta^{(2)}_{n,k}B^H,\end{aligned}$$

where

$$\Delta^{(2)}_{n,k}B^H = B^H_{t^n_{k+1}}-2B^H_{t^n_k}+B^H_{t^n_{k-1}}.$$

Therefore,

$$\begin{aligned}&\sum_{k=1}^{n-1}\left(\Delta^{(2)}_{n,k}D(Y)\right)^2=\sum_{k=1}^{n-1}(\Delta^{(2)}_{n,k}B^H)^2+O_{\omega}(h_n^{\theta+\gamma}) \\ &=h_n^{2H}(4-2^{2H})V^{(2)\widehat{B}^H}_{n,T}+O_{\omega}(h_n^{\theta+\gamma}) \\ &=Th_n^{2H-1}(4-2^{2H})n^{-1}V^{(2)\widehat{B}^H}_{n,T}+O_{\omega}(h_n^{\theta+\gamma}) \\ &=Th_n^{2H-1}(4-2^{2H})[n^{-1}V^{(2)\widehat{B}^H}_{n,T}+O_{\omega}(h_n^{1-2H+\theta+\gamma})],\end{aligned} \tag{6.22}$$

where

$$V^{(2)\widehat{B}^H}_{n,T}:=\frac{1}{(4-2^{2H})h_n^{2H}}\sum_{k=1}^{n-1}(\mathit{\Delta}^{(2)}_{n,k}B^H)^2.$$

Note that

$$1-2H+\theta+\gamma=\begin{cases}1-2H+2(H-\varepsilon)>0, \\ \text{if } \theta=\gamma,\ 0<\varepsilon<1/2; \\ 1-2H+2(H-\varepsilon)>1/2, \\ \text{if } \theta=\gamma,\ 0<\varepsilon<1/4; \\ 1-H-\varepsilon+\beta>0, \\ \text{if } \theta=\beta,\ 0<\beta\leqslant 1; \\ 1-2H+\beta+\gamma>1-2H+2\beta>1/2, \\ \text{if } \theta=\beta, 0\vee(H-1/4)<\beta\leqslant 1.\end{cases} \tag{6.23}$$

We next turn to estimate the strong consistency of the estimator H_n. Note that

$$\begin{aligned}
&\ln\left(\frac{\sum_{k=1}^{2n-1}\left(\Delta_{2n,k}^{(2)}D(Y)\right)^2}{\sum_{k=1}^{n-1}\left(\Delta_{n,k}^{(2)}D(Y)\right)^2}\right)\\
&=\ln\left(\frac{Th_{2n}^{2H-1}(4-2^{2H})[(2n)^{-1}V_{2n,T}^{(2)\widehat{B}^H}+O_\omega(h_{2n}^{1-2H+\theta+\gamma})]}{Th_n^{2H-1}(4-2^{2H})[n^{-1}V_{n,T}^{(2)\widehat{B}^H}+O_\omega(h_n^{1-2H+\theta+\gamma})]}\right)\\
&=(1-2H)\ln 2+\ln\left(\frac{(2n)^{-1}V_{2n,T}^{(2)\widehat{B}^H}+O_\omega(h_{2n}^{1-2H+\theta+\gamma})}{n^{-1}V_{n,T}^{(2)\widehat{B}^H}+O_\omega(h_n^{1-2H+\theta+\gamma})}\right).
\end{aligned}$$

If $0<\beta\leqslant 1$ then $h_n^{1-2H+\theta+\gamma}\to 0$. Moreover, since $n^{-1}V_{n,T}^{(2)\widehat{B}^H}\xrightarrow{a.s.} 1$ (see Appendix A) then

$$\widehat{H}_n=\frac{1}{2}-\frac{1}{2\ln 2}\ln\left(\frac{\sum_{k=1}^{2n-1}\left(\Delta_{2n,k}^{(2)}D(Y)\right)^2}{\sum_{k=1}^{n-1}\left(\Delta_{n,k}^{(2)}D(Y)\right)^2}\right)\xrightarrow{a.s.} H.$$

To estimate the rate of convergence of H_n to H, assume that $0\vee(H-1/4)<\beta\leqslant 1$. From (6.22) and (A.2) in Appendix one gets

$$\begin{aligned}
&h_n^{1-2H}\sum_{k=1}^{n-1}\left(\Delta_{n,k}^{(2)}D(Y)\right)^2\\
&\quad=(4-2^{2H})T[1+O_\omega(n^{-1/2}\ln^{1/2}n)]+O_\omega\Big(\Big(\frac{1}{n}\Big)^{1-2H+\theta+\gamma}\Big)\\
&\quad=(4-2^{2H})T(1+O_\omega(n^{-1/2}\ln^{1/2}n))
\end{aligned}$$

and by Maclaurin's expansion,

$$\begin{aligned}
&\ln\left(\frac{\sum_{k=1}^{2n-1}\left(\Delta^{(2)}_{2n,k}D(Y)\right)^2}{\sum_{k=1}^{n-1}\left(\Delta^{(2)}_{n,k}D(Y)\right)^2}\right)\\
&=\ln\left(\frac{2^{-(2H-1)}(4-2^{2H})T\left[1+O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right)\right]}{(4-2^{2H})T\left[1+O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right)\right]}\right)\\
&=(2H-1)\ln 2^{-1}+\ln\left(\frac{1+O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right)}{1+O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right)}\right)\\
&=(2H-1)\ln 2^{-1}+\ln\left(1+O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right)\right)\\
&=(2H-1)\ln 2^{-1}+O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\widehat{H}_n=&\frac{1}{2}-\frac{1}{2\ln 2}\left[(2H-1)\ln 2^{-1}+O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right)\right]\\
=&\frac{1}{2}+\frac{1}{2}(2H-1)+O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right)=H+O_\omega\left(\left(\frac{\ln n}{n}\right)^{1/2}\right).
\end{aligned}$$

Now we investigate the asymptotic normality of the estimator $\widehat{H}_n$. From (6.22) it follows that

$$\begin{aligned}
\widehat{H}_n=&\frac{1}{2}-\frac{1}{2\ln 2}\ln\left(\frac{(2n)^{-1}V^{(2)\widehat{B}^H}_{2n,T}}{2^{2H-1}n^{-1}V^{(2)\widehat{B}^H}_{n,T}}\left(\frac{1+O_\omega(h_{2n}^{1-2H+\theta+\gamma})}{1+O_\omega(h_n^{1-2H+\theta+\gamma})}\right)\right)\\
=&\widetilde{H}_n-\frac{1}{2\ln 2}\ln\left(1+O_\omega(h_n^{1-2H+\theta+\gamma})\right)=\widetilde{H}_n+O_\omega(h_n^{1-2H+\theta+\gamma}),
\end{aligned}$$

where

$$\widetilde{H}_n=\frac{1}{2}-\frac{1}{2\ln 2}\ln\left(\frac{(2n)^{-1}V^{(2)\widehat{B}^H}_{2n,T}}{2^{2H-1}n^{-1}V^{(2)\widehat{B}^H}_{n,T}}\right)=H-\frac{1}{2\ln 2}\ln\left(\frac{(2n)^{-1}V^{(2)\widehat{B}^H}_{2n,T}}{n^{-1}V^{(2)\widehat{B}^H}_{n,T}}\right).$$

By the limit results of Appendix A we get

$$2\ln 2\sqrt{n}(\widetilde{H}_n - H) \xrightarrow{d} \mathcal{N}(0, \sigma_H^2).$$

Assume that $0 < \beta \leqslant 1$. Now the application of Slutsky's theorem and the results obtained above completes the proof. Note that the limit variance σ_H^2 of $\widehat{H}_n$ equals to that of $\widetilde{H}_n$. $\square$

Chapter 7

Conclusions

Based on the findings presented in the preceding chapters, we draw the following conclusions:

- We gave sufficiently simple conditions under which the solution of SDE (3.1) has a unique positive solution. Furthermore, by applying the chain rule and the inverse Lamperti transform, we proved that there is an injective mapping between the solutions of SDE (3.1) and (3.2) under certain conditions.

- We showed that the solutions of equations (3.1) and (3.2) can be approximated by an implicit Euler approximation method (3.5), which preserves the positivity of the numerical scheme and has the almost sure convergence rate.

- We demonstrated that several classical models (i.e. Ait-Sahalia, Heston) satisfy conditions imposed on SDE (3.2) and hence can be approximated by an implicit Euler approximation method (3.5). Additionally, we discussed the limitations of our approach when applying it to the CKLS model.

- We proposed a Milstein-like approximation scheme of SDE (4.1) and revealed its high convergence rate. Moreover, we showed that this approximation scheme can be applied to the Pearson diffusion model.

- By combining results from number and probability theory we suggested an approximation scheme for integrated fractional Brownian motion with good convergence rate.

- We formed a mathematically rigid framework from a stochastic point of view to describe processes with soft wall condition in the form of fSDE. Subsequently, we proposed and proved the validity of conditions set for fSDE solution existence and uniqueness. This condition set covers a comprehensive class of functions and processes.

- We presented and investigated the process with a soft wall approximation scheme, which has a good convergence rate and is easy to use for process trajectories simulations. This fact was proven and explored by simulating the Vasicek process with soft wall conditions of varying levels of complexity.

- We obtained a strongly consistent and asymptotically normal estimate of the Hurst index for the solution of equations (3.2) and (6.1).

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Chapter 8

Santrauka (Summary in Lithuanian)

8.1 Įvadas

Modeliavimo pagrįsto stochastiniais procesais ir stochastinėmis diferencialinėmis lygtimis (SDL) istorija yra turtinga ir turininga įvairiose srityse. Pastaraisiais dešimtmečiais bandyta išplėsti Vynerio (Wiener) proceso valdomas klasikines stochastines diferencialines lygtis, kurių konstrukcijos pagrindas – Itô stochastinis integralas, į lygtis, kurias valdo trupmeninis Brauno judesys (tBj) ir kurių pagrindas – tBj valdomas integralas (žr., pvz., [47, 48, 58] ir t.t.). Tai nuspręsta padaryti, nes trupmeninis Brauno judesys B^H yra Vynerio proceso W apibendrinimas, į SDL įvedantis naujas trumpalaikės / ilgalaikės atminties savybes (žr. [24, 54]), o tai smarkiai išplečia daugelio finansų, medicinos, biologijos ir fizikos srityse stebimų reiškinių modeliavimo galimybes (žr., pvz., [5, 6, 51, 56]).

Pavyzdžiui, finansų srityje trupmeninių SDL taikymas leidžia tiksliau atvaizduoti turto kainų dinamiką, užfiksuojant finansinėse rinkose stebimą ilgalaikę priklausomybę ir kintamumą. Šis patobulintas modelis pasirodė naudingas statistiniam arbitražui, rinkos efektyvumo tyrimams ekonofizikoje ir portfelių optimizavimui (pvz., [13, 16, 22]).

Taip pat medicinoje ir biologijoje tBj valdomos SDL pritaikytos sudėtingiems fiziologiniams procesams modeliuoti, tokiems kaip ląstelių judėjimas, ligų plitimas ar biomolekulių elgsena. Toks tBj panaudojimas leido atsižvelgti į atminties poveikį įvairiose biologinėse sistemose (pvz., [17, 28, 40]).

Šiame darbe nagrinėjami įvairūs difuzinio proceso, apibrėžto stochastine diferencialine lygtimi

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dZ_s, \qquad t \in [0, T], \tag{8.1}$$

atvejai. Čia $Z = (Z_t)_{t \geq 0}$, $Z_0 = 0$, yra bet koks stochastinis procesas su tolydžiomis trajektorijomis, o $b, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ yra tolydžios funkcijos. Stochastinį integralą (8.1) lygtyje galime interpretuoti, kaip patrajektorį Riemann–Stieltjes integralą arba kaip Itô integralą (priklausomai nuo Z prigimties). Pavyzdžiui, kaip atskirą atvejį galime paimti $Z = B^H$, čia B^H yra tBj.

Nuo difuzinio proceso pavidalo priklauso ir jo tyrimo metodai. Labai svarbu žinoti postūmio koeficiento b, difuzijos koeficiento σ elgesį ir atsitiktinio proceso Z savybes. Juos ištyrus, galima identifikuoti plačias SDL klases, kurios turi panašias matematines struktūras ir savybes. Pavyzdžiui, viena SDL klasė gali turėti tiesinį postūmio koeficientą ir pastovų difuzijos koeficientą, o kita klasė gali turėti netiesinį postūmį ir nuo laiko priklausomą difuzijos koeficientą.

Apibrėžę klases, toliau galime tirti ir analizuoti konkretesnius kiekvienos klasės poklasius. Šie poklasiai dažniausiai vadinami modeliais, kurie detaliau fiksuoja specifines charakteristikas ir prielaidas apie tiriamą sistemą. Žinoma, reikėtų pastebėti, kad klasės ir modelio skirtumai nėra griežtai apibrėžti ir tai yra labiau praktinės sąvokos, palengvinančios mūsų samprotavimus.

Stochastinių modelių istorija ir klasifikacija yra labai plati ir išsami, šiame darbe nagrinėjamas tik keletas iš jų. Konkrečiai, Chan-Karolyi-Longstaff-Sanders (CKLS) modelis, naudojamas aproksimuojant obligacijų ir obligacijų opcijų kainas, valiutos keitimo kursus ar draudimines pretenzijas (pvz., [8, 49, 53]); Ornstein-Ulenbeck ir Vasicek procesai, taikomi nuo fenotipinių požymių evoliucijos analizės iki palūkanų normų (pvz., [4, 21]); Pearson difuzijos modelis, glaudžiai susijęs su Fokker-Planck lygtimi ir taikomas fizikoje, chemijoje, inžinerijoje, reologijoje, aplinkos moksluose bei finansų matematikoje (pvz., [39, 50]).

Galiausiai, vienas iš naujesnių aspektų stochastinių diferencialinių lygčių tyrimuose, atvėręs naujų tyrimo klodų mūsų darbui, yra geometriškai apribotų erdvių, kuriose sklinda procesas, įtaka tBj ir SDL savybėms [20]. Klasikiniu atveju dauguma pirmiau aptartų SDL klasių ir modelių „juda laisvai“ (t. y. nėra kažkaip ribojami aplinkos). Nors tokia prielaida yra pagrįsta ir logiška finansų srityje, gamtos moksluose taip dažniausiai nėra (pvz., norint modeliuoti serotonerginių aksonų augimą smegenų kamiene [27, 55, 56] ar angiogenezę navike [7] procesas yra ribojamas aplinkinių audinių). Vienas iš būdų, leidžiančių sumodeliuoti aplinkos pasipriešinimo reiškinius, kuris ir tyrinėjamas šioje disertacijoje, yra sienos sąlygų/dedamųjų įvedimas į pradinę SDL. Įdomu, kad egzistuoja visas tokių sąlygų su skirtingomis savybėmis (pvz., elastinga siena, neelastinga siena, lipni siena [55]) spektras.

Dar daugiau, galima nubrėžti tiesioginį ryšį tarp šių SDL su sienomis ir SDL su teigiamais sprendiniais [3, 10, 19, 46, 52, 57]. Pakanka teigiamumo sąlygą interpretuoti, kaip ribojimus atsitiktinio proceso judėjimui pusiau begaline ašimi ($X \in (0, \infty)$ arba $X \in [0, \infty)$) dėl atspindinčios sienos taške 0. Taigi, galime sakyti, kad visus šiame darbe pateiktus tyrimus vienija ribų stochastinėms diferencialinėms lygtims idėja.

8.2 Tikslai ir uždaviniai

Trumpai apžvelgsime šio darbo tikslus ir uždavinius.

3 skyriuje tyrinėjame trupmeninių stochastinių diferencialinių lygčių (tSDL) klasę su koeficientais, kurie gali netenkinti tiesinio augimo sąlygos ar neturėti difuzijos koeficiento, tenkinančio Lipšico sąlygą. Mūsų tikslas – rasti sąlygas, užtikrinančias tokių tSDL sprendinių teigiamumą, ir pademonstruoti šiems sprendiniams pritaikytų atbulinio Oilerio metodo aproksimacijų konvergavimą. Be to, kadangi bet kuriai tSDL Hursto indeksas yra vienas iš esminių parametrų, norime gauti kuo aukštesnės „kokybės“ šio parametro įvertį teigiamiems tSDL sprendiniams. Galiausiai, kad mūsų rezultatai būtų praktiškai pritaikomi, pageidautina rasti konkrečius klasikinių modelių pavyzdžius, tenkinančius mūsų teoremų sąlygas.

4 skyriuje tyrinėjame vienmates SDL, valdomas stochastinio proceso su Hiolderio prasme tolydžiomis trajektorijomis, kurių eilė $1/2 < \gamma < 1$. Kadangi tokių SDL sprendinio išreikštinė forma nėra žinoma, reikia sukurti sprendinio aproksimavimo schemą ir rasti jos konvergavimo greitį. Žinoma, norint, kad ši nauja aproksimavimo schema būtų vertinga, jos konvergavimo greitis turi būti didesnis už kitų autorių ankstesniuose tyrimuose pasiektus greičius. Be to, 5 skyriuje taip pat tyrinėjame integruotą trupmeninį Brauno judesį ir ieškome jo aproksimavimo schemos.

Vėlgi, kad rezultatai būtų praktiškai pritaikomi, reikia rasti klasikinių modelių pavyzdžius, patenkančius į SDL klasę, tyrinėjamą 4 skyriuje.

6 skyriuje pristatome trupmeninę SDL su minkštąja sienele. Pastebėta, kad SDL su atspindžiu gali būti interpretuojamos kaip lygtys su kietąja sienele, o vietoje atspindžio įvedę atstūmimą, gauname SDL su minkštąja sienele. Priešingai nei SDL su atspindžiu, kuriame procesas negali peržengti kietosios sienelės, minkštoji sienelė yra atstumianti, bet nėra nepralaidi. Peržengęs minkštosios sienelės ribą, procesas patiria tam tikro dydžio jėgą priešinga kryptimi (kai procesas yra toli nuo sienelės, jėga veikia silpnai). Mūsų darbe tiriamas modelis turi šias pirmiau išvardytas savybes. Lygčiai, apibrėžiančiai tiriamą modelį, randame sąlygas, kai ji turi vienintelį sprendinį, ir sukonstruojame neišreikštinę Oilerio aproksimavimo schemą bei nustatome konvergavimo greitį. Be to, dėl modelio naujumo svarbu pademonstruoti jo elgsenos ypatumus naudojant matematinio modeliavimo priemones.

8.3 Metodai

Šiame darbe taikome įvairius stochastinių diferencialinių lygčių tyrimuose naudojamus metodus.

Kadangi difuzijos koeficiento sukeliama sąveika tarp atsitiktinio triukšmo ir pačio SDL proceso (3 ir 4 skyriuose) paprastai apsunkina lygties savybių tyrimą, naudojame Lamperti transformaciją, kuri pakeičia (suveda) nagrinėjamą SDL į lygtį su pastoviu difuzijos koeficientu, o tai leidžia išvengti šios sąveikos. Žinoma, norint panaudoti tokią transformaciją, reikėjo nustatyti tam tikras sąlygas difuzijos komponentui ir pademonstruoti pradinės SDL sprendinio, papildomo sprendinio ir jų aproksimacijų suderinamumą.

Visos trys SDL klasės, tyrinėjamos 3, 4 ir 6 skyriuose, neturi sprendinių, kurie galėtų būti užrašyti išreikštine forma. Todėl teko sukurti įvairias aproksimavimo schemas. Šios schemos varijuoja nuo klasikinių išreikštinių ir neišreikštinių Oilerio tipo schemų iki originalesnių, panašių į Milšteino tipo schemas.

6 skyriuje, įrodinėjant sprendinio egzistavimą SDL klasei su minkštąja sienele, taikėme neišreikštinį Pikaro iteracijų metodą. Šis metodas pasirodė tinkamas labiausiai, nes neįveda naujų apribojimų atstūmimo jėgai SDL, palyginus su kitais metodais (pvz., išreikštiniu Pikaro itera-

cijų metodu, žr. [32]).

Integruoto trupmeninio Brauno judesio aproksimavimo schemos asimptotika 5 skyriuje rasta derinant tikimybių ir skaičių teorijos rezultatus. Konkrečiai, pritaikyti tam tikri rezultatai apie harmoninius skaičius, kurie (kiek mums žinoma) anksčiau nebuvo taikyti SDL tyrimuose.

Hursto indekso H įverčiai SDL trajektorijoms 3 ir 6 skyriuose atlikti naudojant schemas, pagrįstas antros eilės pokyčiais, kvadratinėmis variacijomis ir jų ribiniu elgesiu.

8.4 Aktualumas ir naujumas

Šiame darbe pateikti rezultatai yra nauji, originalūs ir iš viso atitinka keturis straipsnius pripažintuose matematikos žurnaluose [35, 36, 37, 44]. Šie rezultatai išplečia ir patobulina naujausius kitų autorių darbus, tiriant tSDL klases, apimdami platesnį stochastinių modelių spektrą ir pasiūlydami naujas aproksimavimo schemas, kurios turi didesnius konvergavimo greičius šių tSDL sprendiniams. Be to, kai kuriuose mūsų tyrimuose sukonstravome ir tyrinėjome naujas tSDL, kurios gali būti itin įdomios ne tik matematikams, bet ir kitose srityse dirbantiems mokslininkams (pvz., biologijoje, medicinoje ar fizikoje).

8.5 Tyrimų istorija

8.5.1 Trupmeninis Brauno judesys

Anot Kubiliaus [38], Tudoro [54] ir kt., pirmasis atsitiktinį procesą su trupmeninio Brauno judesio savybėmis apibrėžė A. N. Kolmogorovas [31] 1940 metais. Įdomu, kad Kolmogorovas šiuos procesus vadino ne trupmeniniu Brauno judesiu, o Vynerio spiralėmis (tuo metu, stochastinei analizei dar tik žengiant pirmuosius savo žingsnius, jo požiūriui didelę įtaką turėjo geometrija). Na, o pirmasis fundamentalus tyrimas, skirtas tBm, kuris ir suteikė šiam procesui jo pavadinimą (t. y. trupmeninis Brauno judesys), publikuotas Benoit Mandelbrot ir John Ness 1968 metais [43].

1 apibrėžimas. *Trupmeniniu Brauno judesiu su Hursto indeksu* $H \in (0,1)$ *vadinsime Gauso procesą* $B^H = \{B_t^H, t \in \mathbb{R}^+\}$*, turintį savybes:*

(i) $B_0^H = 0$;

(ii) $\mathbb{E}B_t^H = 0, \quad t \in \mathbb{R}^+$;

(iii) $\mathbb{E}B_t^H B_s^H = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \in \mathbb{R}^+$.

Pasinaudojus šiomis fundamentaliosiomis tBj savybėmis galime įrodyti, kad

1 teiginys. *Jei B^H yra trupmeninis Brauno judesys, tai*

1. *(Savipanašumas)* $\forall a > 0, \quad B_{at}^H \overset{D}{=} a^H B_t^H$;
2. *(Pokyčių stacionarumas)* $\forall h > 0, \quad B_{t+h}^H - B_t^H \overset{D}{=} B_h^H$;
3. *(Reguliarumas)* $\forall \epsilon > 0, \exists a.d.\ C_\epsilon : |B_t^H - B_s^H| \leq C_\epsilon |t-s|^{H-\epsilon}$.

Trupmeninio Brauno judesio 3 reguliarumo savybė gali būti išplėsta į dvi teoremas:

1 teorema (tolydumas Hiolderio prasme (žr. [38], p. 4)**.** *Beveik visos trupmeninio Brauno judesio trajektorijos yra lokaliai Hiolderio su eile griežtai mažesne už $H \in (0,1)$. T. y. kiekvienam $T > 0$ egzistuoja neneigiamas atsitiktinis dydis $G_{\gamma,T}$ toks, kad $\mathbb{E}(|G_{\gamma,T}|^p) < \infty$, kiekvienam $p \geqslant 1$ ir*

$$|B_t^H - B_s^H| \leqslant G_{\gamma,T}|t-s|^\gamma \quad b.v. \tag{8.2}$$

visiems $s,t \in [0,T]$, čia $\gamma \in (0,H)$.

2 teorema (B^H imties modulis (žr. [18], p. 48))**.** *Funkcija $\rho_H(u) = u^H\sqrt{|\ln u|}$, kiekvienam $u \geqslant 0$ yra B^H imties modulis, t. y. beveik visiems $\omega \in \Omega$ egzistuoja $K_\omega < \infty$ toks, kad*

$$|B_t^H(\omega) - B_s^H(\omega)| \leqslant K_\omega \rho_H(|t-s|), \quad \text{čia } t,s \in [0,T]. \tag{8.3}$$

Vynerio procesas *versus* trupmeninis Brauno judesys

Pastebime, kad kai Hursto indeksas įgyja reikšmę $H = \frac{1}{2}$, trupmeninio Brauno judesio 1 apibrėžimas tampa Vynerio proceso (dažniausiai žymimo W) apibrėžimu (pokyčių nepriklausomumą galima iš šio apibrėžimo pademonstruoti gan paprastai).

2 apibrėžimas. *Tegul $W = \{W_t, t \geq 0\}$ yra stochastinis procesas, įgyjantis realiąsias reikšmes. Sakysime, kad W yra Vynerio procesas (standartinis Brauno judesys), jei tenkina šias savybes:*

(i) $W_0 = 0$;

(ii) Proceso W pokyčiai yra nepriklausomi;

(iii) Pokyčiai $W_t - W_s$ yra pasiskirstę pagal Gauso dėsnį su vidurkiu 0 *ir dispersija $t-s$ (t. y. $W_t - W_s \sim \mathcal{N}(0, t-s)$).*

Iš pirmo žvilgsnio, gali pasirodyti, kad ryšys tarp Vynerio proceso W ir trupmeninio Brauno judesio B^H yra labai tamprus ir natūralus. Tačiau atidžiau įsižiūrėjus paaiškėja, kad toks teiginys nėra teisingas. Pasirodo, kad skirtumas tarp Vynerio proceso pokyčių, kurie yra nepriklausomi ir trupmeninio Brauno judesio pokyčių, kurie yra priklausomi, yra esminis. Ne tik dauguma teorinių ir praktinių metodų, taikomų Vynerio proceso tyrimuose, negali būti tiesiogiai taikomi trupmeniniam Brauno judesiui, bet šis iš pažiūros nedidelis skirtumas iš esmės pakeičia pačią procesų prigimtį.

Vienas iš geriausių šių procesų prigimties skirtumo pavyzdžių yra atminties reiškinys, kuriuo pasižymi tik trupmeninis Brauno judesys. Yra daug įvairių būdų apibrėžti, kas yra proceso atmintis, bet bendrai galima pasakyti, kad procesas turi atmintį, jei jis yra ne Markovo (t. y. ne visa informacija apie proceso elgseną yra sukaupta dabartyje). Todėl

3 apibrėžimas. *Sakysime, kad procesas X turi ilgalaikę atmintį, jei*

$$\sum_{i \geq 0} r_i = \infty$$

ir turi trumpalaikę atmintį, jei

$$\sum_{i \geq 0} r_i < \infty,$$

čia $r_i = \mathbb{E}(X_1 - X_0)(X_{i+1} - X_i)$.

Kadangi trupmeniniam Brauno judesiui turime

$$r_i = \frac{1}{2}\left(\left((i+1)^{2H} - i^{2H}\right) - \left(i^{2H} - (i-1)^{2H}\right)\right), \quad i \geq 1, \quad r_0 = 1,$$

tai pritaikius teleskopavimą gausime $2\sum_{i=0}^{n} r_i = (n+1)^{2H} - n^{2H} + 1$, kuri pakankamai dideliems n elgiasi, kaip $(n^{2H})' = 2Hn^{2H-1}$, t. y. konverguoja, kai $H \leq \frac{1}{2}$, ir diverguoja, kai $H > \frac{1}{2}$. Atitinkamai, trupmeninis

Brauno judesys turi ilgalaikę atmintį, kai $H > \frac{1}{2}$, ir trumpalaikę atmintį, kai $H \leq \frac{1}{2}$.

Reikėtų pastebėti, kad 3 apibrėžimas gali būti šiek tiek klaidinantis. Nors Vynerio procesui $\sum_{i \geq 0} r_i < \infty$, bet taip nutinka dėl to, kad Vynerio procesas neturi atminties apskritai (jo pokyčiai nepriklausomi).

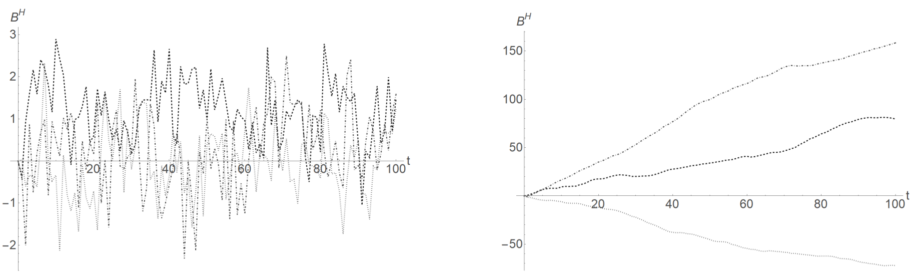
8.5.2 Praktiniai Hursto indekso poveikio aspektai

Analizuojant Hursto indekso įtaką trupmeninio Brauno judesio trajektorijoms, galima išskirti du pagrindinius būdus šiai įtakai apibūdinti.

Pirmasis gali būti priskirtas dar paties britų hidrologo Haroldo Edvino Hursto darbams [24], t. y. Hursto indekso H įtakai B_t^H trajektorijų vystymosi linkmei (ypač gerai matomai „ilgiems“ laiko intervalams):

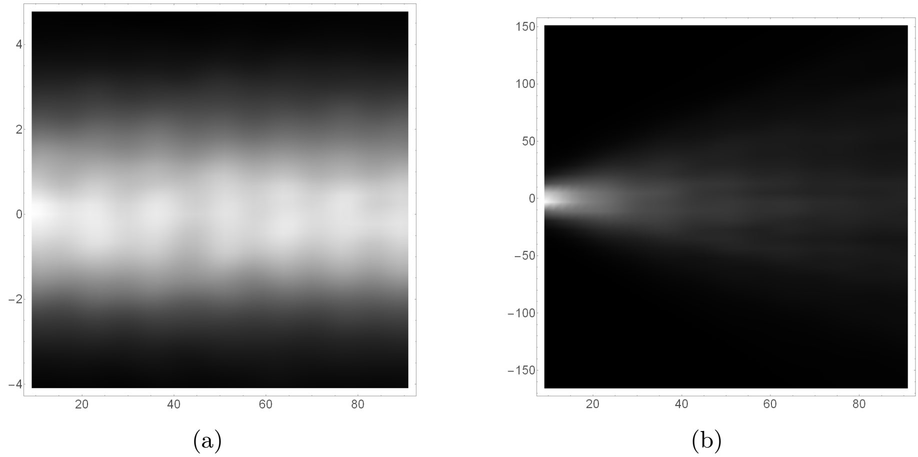
1. $H < \frac{1}{2}$ – svyravimo aplink vidurkį (antitendencinga) elgsena, kai teigiamą / neigiamą proceso vertės pokytį su didžiausia tikimybe seka atitinkamai neigiamas / teigiamas proceso vertės pokytis. Todėl proceso trajektorijos svyruoja aplink vidurkį $\mathbb{E}B^H$ (žr. 8.1(a) pav. ir 8.2(a) pav.).

2. $H > \frac{1}{2}$ – pastovi (tendencinga) elgsena, kai teigiamą / neigiamą proceso vertės pokytį su didžiausia tikimybe seka taip pat atitinkamai teigiamas / neigiamas proceso vertės pokytis. Todėl proceso trajektorijos yra linkusios nukrypti nuo vidurkio $\mathbb{E}B^H$ (žr. 8.1(b) pav. ir 8.2(b) pav.).



(a) (b)

8.1 pav. Trupmeninio Brauno judesio trajektorijų realizacijos.
(**a**) Hursto indeksas $H = 0,05$. (**b**) Hursto indeksas $H = 0,95$.



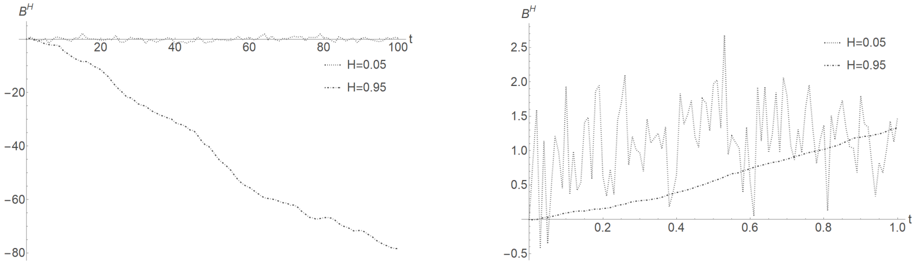
8.2 pav. 1000 trupmeninio Brauno judesio trajektorijų karščio žemėlapis. (**a**) Hursto indeksas $H = 0,05$. (**b**) Hursto indeksas $H = 0,95$.

Antrasis būdas pažvelgti į Hursto indeksą susijęs labiau su trupmeninio Brauno judesio trajektorijų nereguliarumu ir „nervingumu“, kurį sukelia skirtingos Hursto indekso reikšmės, ypač „trumpiems“ laiko intervalams, nes

$$B^H_{t^n_{k+1}} - B^H_{t^n_k} \overset{D}{=} \frac{T^H}{n^H} B^H_1, \tag{8.4}$$

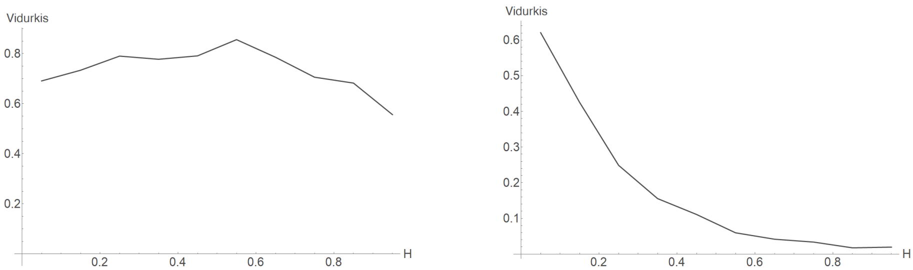
čia $\pi^n = \{t^n_k = (k/n)T, 1 \le k \le n\}$ yra tolygaus intervalo $[0,T]$ skaidinio seka.

Pastebime, kad (8.4) lygtyje prieš Brauno judesį esantis narys n^{-H} turi slopinantį eksponentinį poveikį, kai H didėja. Todėl tBm trajektorijų nepastovumas „trumpiems“ laiko intervalams 8.3(b) paveikslėlyje yra daug labiau išreikštas, nei ilgiems laiko intervalams 8.3(a) paveikslėlyje. Kad tai nėra optinė iliuzija, galima įrodyti, palyginus pokyčių reikšmių modulių vidurkius su Hursto indeksu (8.4 pav.). Matome, priklausomybė yra aiškiai eksponentinė (kaip tikėtasi iš (8.4) lygties) „trumpiems“ laiko intervalams (8.4(b) pav.) ir nepastebima arba net neegzistuojanti „ilgiems“ laiko intervalams (8.4(a) pav.).



(a) (b)

8.3 pav. TBj trajektorijų palyginimas. (**a**) „Ilgi“ laiko intervalai. (**b**) „Trumpi“ laiko intervalai.



(a) (b)

8.4 pav. Pokyčių modulių vidurkis nuo Hursto indekso. (**a**) „Ilgiems“ laiko intervalams. (**b**) „Trumpiems“ laiko intervalams.

8.5.3 Integravimas patrajektoriui trupmeninio Brauno judesio atžvilgiu

Žinoma, kitas žingsnis analizuojant trupmeninį Brauno judesį – integravimo stochastinių procesų atžvilgiu uždavinys. Yra ne vienas būdas tai padaryti trupmeninio Brauno judesio atžvilgiu, tačiau mūsų tyrimuose bus naudojamas metodas, grindžiamas Riemann-Stieltjes integralu ir p-variacijomis:

4 apibrėžimas. *Tegul* $0 < p < \infty$. *Tada funkcijos* $f : [a.b] \to \mathbb{R}$ *p-variacija vadinsime*

$$v_p(f,[a,b]) := \sup_{\tau} \left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p \colon t_i \in \tau \right\},$$

čia τ *yra betkoks intervalo* $[a,b]$ *skaidinys.*

Atkreipsime skaitytojo dėmesį, kad p variacijos gali būti begalinės. Taigi sakysime, kad funkcija f turi baigtinę p variaciją, jei

$$v_p(f,[a,b]) < \infty.$$

5 apibrėžimas. *Tegul* $\tau^n = \{a = t_0^n < t_1^n < ... < t_{k_n}^n = b\}, n \in \mathbb{N}$, *yra skaidinių seka tokia, kad* $\max_i |t_{i+1}^n - t_i^n| \to 0$,
o $\xi^n = \{\xi_i^n \in [t_i^n, t_{i+1}^n], i = 0,1,...,k_n - 1\}, n \in \mathbb{N}$, *bet kokia tarpinių skaidinių seka. Tada, funkcijos f Riemann-Stieltjes integralu* $\int_a^b f(t)dg(t)$ *funkcijos g atžvilgiu intervale* $[a,b]$ *vadinsime ribą*

$$\int_a^b f(t)dg(t) := \lim_{n\to\infty} \sum_{i=0}^{k_n-1} f(\xi_i^n)(g(t_{i+1}^n) - g(t_i^n)), \tag{8.5}$$

jei ši riba egzistuoja ir yra nepriklausoma nuo skaidinių τ^n *ir* ξ^n.

Pažymėkime $\mathcal{W}([a,b])$ aprėžtų p variacijos funkcijų klasę intervale $[a,b]$. Tada

3 teorema (Young-Stieltjes integruojamumo teorema [59])**.** *Tegul*
$f \in \mathcal{W}_p([a,b])$ *ir* $g \in \mathcal{W}_q([a,b])$, *čia* $p,q > 0 : 1/p + 1/q > 1$. *Jei f ir g neturi bendrų trūkio taškų, tai Riemann-Stieltjes integralas* $\int_a^b f dg$ *egzistuoja ir kiekvienam* $\xi \in [a,b]$ *galioja nelygybė:*

$$\left|\int_a^b f dg - f(\xi)[g(b) - g(a)]\right| \le \left(1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right) V_p(f;[a,b]) V_q(g;[a,b]),$$

čia $\zeta(s) := \sum_{n\ge1} n^{-s}$ *yra Riemann dzeta funkcija ir*

$$V_p(\cdot;[a,b]) = v_p^{1/p}(\cdot;[a,b]).$$

Pasinaudojus tBj reguliarumo savybe galima nesunkiai įrodyti, kad tBj turi baigtinę p-variaciją b.v., kai $p > 1/H$. Taigi, 3 teoremą galima pritaikyti Riemann-Stieltjes integralams trupmeninio Brauno judesio atžvilgiu.

8.5.4 Stochastinės diferencialinės lygtys

Apibrėžus trupmeninį Brauno judesį ir stochastinį integravimą (patrajektoriui), galime pradėti tirti integralines lygtis, kurias galima užrašyti forma:

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dZ_s, \qquad t \in [0, T], \tag{8.6}$$

čia $Z = (Z_t)_{t \geqslant 0}$, $Z_0 = 0$ yra bet koks atsitiktinis procesas, su tolydžiomis trajektorijomis, o $b, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ yra kažkokios tolydžios funkcijos.

Dažnai (iš dalies dėl istorinių priežasčių) lygtį (8.6) rasime užrašytą forma:

$$dX_t = x_0 + b(t, X_t)\, dt + \sigma(t, X_t)\, dZ_t, \qquad X_0 = x_0. \tag{8.7}$$

Savaime suprantama, kad abiejų formų (8.6) ir (8.7) lygtys yra ekvivalenčios. Paprastai funkcija b vadinama postūmio koeficientu, o funkcija σ – difuzijos koeficientu.

SDL sprendinys

Natūralu, kad vienas iš svarbiausių ir esminių SDL uždavinių yra SDL sprendinių radimas ir analizė.

6 apibrėžimas. *Tolydų atsitiktinį procesą $X_t, t \in [0, T]$ vadinsime (8.7) SDE sprendiniu intervale $[0, T]$, jei su tikimybe 1 tenkinama (8.6) kiekvienam $t \in [0, T]$.*

Dažnai (8.6) lygties sprendinius vadinsime difuzijos procesais.

SDL sprendinio egzistavimas ir vienatis dažniausiai priklauso nuo funkcijų b, σ, savybių ir valdančiojo atsitiktinio proceso Z tipo. Istoriniu požiūriu didžiausia pažanga pasiekta tyrinėjant SDL sprendinius, kai Z yra Vynerio procesas W. Reikėtų pastebėti, kad kaip taisyklė, egzistavimo ir vienaties sąlygos, taikomos Vynerio proceso SDL, yra netinkamos trupmeninio Brauno judesio atveju, o trupmeninių SDL egzistavimo ir vienaties sąlygų rinkiniai dažniausiai yra labiau ribojantys ir veikia siauresnėms funkcijų b, σ klasėms.

8.5.5 Aproksimacinės schemos

Svarbu pabrėžti, kad sprendinio egzistavimo ir vienaties įrodymas neužtikrina galimybės / gebėjimo užrašyti šį sprendinį išreikštine forma. Labai dažnai tai tiesiog neįmanoma. Be to, net turint išreikštinį sprendinį gali tekti įvertinti Riemann-Stieltjes integralą, kuris (išskyrus trivialius atvejus) įveda savo aproksimacijas. Todėl beveik visada reikia naudoti kažkokius aproksimavimo metodus. Konkrečiai šiame darbe, mus domins diskrečios tolydžių SDL sprendinių aproksimacijos, kurios turbūt yra populiariausios ir plačiausiai naudojamos šioje srityje.

Euler-Maruyama aproksimacijų šeima

Viena iš seniausių diskrečių aproksimavimo schemų yra vadinamoji Euler-Maruyama aproksimavimo metodų šeima. Pradėkime nuo išreikštinio tiesioginio Euler-Maruyama metodo apibrėžimo.

7 apibrėžimas. *Tiesiogine (8.7) SDL sprendinio X Euler-Maruyama aproksimacija vadinsime tolydų stochastinį procesą Y tenkinantį iteracinę schemą:*

$$Y_{t_{i+1}} = Y_{t_i} + b(t_i, Y_{t_i})(t_{i+1} - t_i) + \sigma(t_i, Y_{t_i})(Z_{t_{i+1}} - Z_{t_i}),$$

čia $t_i \in [0,T] : i = 0,...,n-1$.

Dažniausiai naudosime homogeninį laiko skaidinį
(t. y. $\Delta t_i = t_{i+1} - t_i = 1/n$).

Deja, nors ir išreikštinis, tiesioginis Euler-Maruyama metodas nėra visada pageidautinas dėl stabilumo problemų, kurios atsiranda tam tikriems laiko padalijimams. Todėl dažnai naudojamas neišreikštinis atbulinis Euler-Maruyama metodas.

8 apibrėžimas. *Atbuline (8.7) SDL sprendinio X Euler-Maruyama aproksimacija vadinsime tolydų stochastinį procesą Y tenkinantį iteracinę schemą:*

$$Y_{t_{i+1}} = Y_{t_i} + b(t_{i+1}, Y_{t_{i+1}})(t_{i+1} - t_i) + \sigma(t_i, Y_{t_i})(Z_{t_{i+1}} - Z_{t_i}), \qquad (8.8)$$

čia $t_i \in [0,T] : i = 0,...,n-1$.

Lamperti transformacija

Atidžiau pažvelgę į (8.6) ir (8.7) lygtis, pastebėsime, kad difuzijos koeficientas σ gali būti tiek konstanta (pvz., Vasiceko modelių šeima), tiek daug sudėtingesnė funkcija, priklausanti nuo laiko ar atsitiktinio proceso X (pvz., CKLS modelių šeima). Šiuo, antruoju atveju atsiranda sąveika tarp proceso būsenos ir atsitiktinio triukšmo komponentės [14]. Tokia sąveika retai yra pageidautina, nes labai apsunkina sprendinio stabilumo ir aproksimacinių schemų apibrėžtumo užtikrinimą.

Kai kurie metodai, kuriais siekiama išvengti šios sąveikos (su netrivialiu difuzijos koeficientu) sukeltų problemų, siūlo ją tiesiogiai transformuoti naudojant kitas funkcijas [15], riboti laiko padalijimus arba net iš naujo apibrėžti visą aproksimavimo schemą [2]. Tačiau visi šie metodai yra gana grubūs, ribojantys pačių schemų konvergavimo greitį ir apskritai nėra universalūs.

Todėl vienas iš efektyviausių būdų išvengti tokių netrivialių difuzijos koeficientų – visiškai jų atsisakyti. Pavyzdžiui, tai galima pasiekti naudojant Lamperti transformaciją:

4 teorema. *Tegul*

$$X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s^H, \qquad t \in [0,T], \tag{8.9}$$

su griežtai neigiamu / teigiamu difuzijos koeficientu σ.

Vienmatę Lamperti transformaciją apibrėšime, kaip funkciją:

$$Y_t = F(X_t) = \int_0^{X_t} \frac{1}{\sigma(x)}dx, \quad Y_0 = F(X_0). \tag{8.10}$$

Tada Y_t tenkina lygtį:

$$Y_t = Y_0 + \int_0^t \frac{b(F^{-1}(Y_s))}{\sigma(F^{-1}(Y_s))}ds + B_t^H.$$

Šią transformaciją ir naudosime savo darbe, kai susidursime su netrivialiu difuzijos koeficientu.

8.6 Rezultatai

Šiame skyrelyje pristatysime disertacijoje gautus / įrodytus rezultatus bei trumpai aptarsime jų reikšmę ir vertingumą.

8.6.1 Teigiami trupmeninių SDL sprendiniai su difuzijos koeficientu netenkinančiu Lipšico sąlygos

Sprendinio teigiamumas yra esminė savybė įvairiuose finansiniuose modeliuose, įskaitant opcijų kainodarą, stochastinį volatilumą ar palūkanų normų modelius. Todėl ypač svarbu rasti sąlygas, užtikrinančias, kad sprendiniai išliktų teigiami. Be to, kai griežtai teigiamas sprendinys neturi išreikštinio pavidalo ir ieškomos aproksimacijos, būtina užtikrinti pačių aproksimacijų teigiamumą, nes antraip nebus išsaugotas aproksimacinės schemos korektiškumas. Kad egzistuotų vienintelis SDL sprendinys, lygties koeficientai dažniausiai turi tenkinti tiesinio augimo ir lokalaus Lipšico sąlygas. Tačiau šios sąlygos nėra tenkinamos CKLS, CIR, Ait-Sahalia ir daugybės kitų plačiai naudojamų modelių.

Mūsų darbo pagrindinis tikslas buvo rasti sąlygas, kurias turi tenkinti lygtis

$$X_t = x_0 + \int_0^t X_s^\beta f(X_s^{1-\beta})\,ds + \sigma \int_0^t X_s^\beta\,dB_s^H, \qquad \beta > 1,\ H \in (1/2, 1), \tag{8.11}$$

norint užtikrinti šios SDL sprendinio teigiamumą. Stochastinis integralas yra patrajektorinis Riemann–Stieltjes integralas.

Sąlygos

Tegul funkcija f (8.11) lygtyje tenkina šias sąlygas:

($\mathbf{C}_1$) f yra lokaliai Lipšico intervale $(0, +\infty)$;

($\mathbf{C}_2$) egzistuoja konstantos $a > 0$ ir $\alpha \geqslant 0$ tokios, kad

$$\widehat{f}(x) := -f(x) \geqslant \frac{a}{x^{1+\alpha}}$$

kiekvienam pakankamai mažam $x \in (0, \infty)$;

($\mathbf{C}_3$) funkcija $\widehat{f}(x)$ tenkina *vienpusę Lipšico sąlygą*, t. y., egzistuoja konstanta $K \in \mathbb{R}$ tokia, kad

$$(x-y)(\widehat{f}(x) - \widehat{f}(y)) \leqslant K(x-y)^2$$

visiems $x, y \in (0, +\infty)$.

Tegul $\mathcal{C}^\gamma([0,T])$ Hiolderio prasme tolydžių funkcijų erdvė, kurių eilė $\gamma > 0$ intervale $[0,T]$. Tada

5 teorema. *Jei funkcija f tenkina sąlygas ($\boldsymbol{C}_1$)–($\boldsymbol{C}_3$), tada (8.11) lygtis yra korektiška ir turi vienintelį teigiamą $\gamma \in (\frac{1}{2}, H)$ eilės sprendinį $X \in \mathcal{C}^\gamma([0,T])$, kai $H \in (\frac{1}{2}, 1)$.*

Aproksimacinė schema

Mūsų tyrinėjamas SDL sprendinys paprastai neturi išreikštinio pavidalo, todėl turime sukonstruoti jo aproksimacinę schemą ir įrodyti jos korektiškumą.

Pirmiausia, nagrinėkime stochastinę diferencialinę lygtį

$$Y_t = y_0 + (\beta - 1)\int_0^t \widehat{f}(Y_s)\,ds - (\beta - 1)\sigma B_t^H,\; y_0 = x_0^{1-\beta}, t \geqslant 0, H \in (\tfrac{1}{2}, 1). \tag{8.12}$$

Antra, intervalo $[0,T]$ tolygųjį skaidinį pažymėkime $\pi^n = \{t_k^n = \frac{k}{n}T, 1 \leqslant k \leqslant n\}$ ir $h = t_k^n - t_{k-1}^n$, $1 \leqslant k \leqslant n$.

Trečia, apibrėžkime atbulinę Y aproksimacinę schemą:

$$Y_{n,k+1} = Y_{n,k} + (\beta - 1)\widehat{f}(Y_{n,k+1})h - \sigma(\beta - 1)(B^H_{t^n_{k+1}} - B^H_{t^n_k}), \tag{8.13}$$

čia $Y_{n,0} = y_0, \quad 0 \leqslant k \leqslant n-1$.

Galiausiai įveskime papildomą atbulinės Oilerio schemos teigiamumą užtikrinančią sąlygą:

($\mathbf{C}_4$) tegul $\widehat{F}(x) = x - (\beta - 1)\widehat{f}(x)h, x \in (0,\infty)$, čia funkcija $\widehat{f}(x)$ tenkina sąlygą ($\mathbf{C}_3$) bei egzistuoja $h_0 > 0$ toks, kad $\lim\limits_{x\to+\infty} \widehat{F}(x) = +\infty$ ir $\lim\limits_{x\to 0^+} \widehat{F}(x) = -\infty$, kai $0 < h < h_0$.

Tada galime įrodyti

6 teorema. *Tegul f funkcija (8.11) lygtyje yra tolydžiai diferencijuojama intervale $(0,+\infty)$ ir tenkina sąlygą ($\boldsymbol{C}_2$) bei egzistuoja konstanta $K \in \mathbb{R}$ tokia, kad jos išvestinė yra aprėžta iš viršaus, t. y. $f'(x) \leqslant K$. Jei tolygių skaidinių seka π^n intervale $[0,T]$ yra tokia, kad $h < h_0$, tada kiekvienam $T > 0$ ir $H \in (\frac{1}{2}, 1)$,*

$$\sup_{0 \leqslant t \leqslant T} |Y_t - Y_t^n| = O_\omega(n^{-H}\sqrt{\ln n}), \tag{8.14}$$

čia

$$Y_t^n = Y_{n,k} + \frac{t - t_k^n}{h}(Y_{n,k+1} - Y_{n,k})$$

ir $t \in (t_k^n, t_{k+1}^n]$, $k = 0, \ldots, n-1, Y_{n,0}^n = y_0$. Be to,

$$\sup_{0 \leqslant t \leqslant T} |X_t - (Y_t^n)^{-1/(\beta-1)}| = \begin{cases} O_\omega(n^{-H}\sqrt{\ln n}) & \textit{for } \beta \in (1,2], \\ O_\omega((n^{-H}\sqrt{\ln n})^{1/(\beta-1)}) & \textit{for } \beta > 2, \end{cases} \tag{8.15}$$

čia X yra (8.11) *lygties sprendinys.*

9 apibrėžimas. *Tegul* (Z_n) *yra a.d. sekų seka, o* ς *tegul yra b.v. neneigiamas a.d. ir tegul* $(a_n) \subset (0,\infty)$ *yra nykstanti seka. Tada* $Z_n = O_\omega(a_n)$ *reiškia, kad* $|Z_n| \leq \varsigma \cdot a_n$ *kiekvienam* n, *o* $Z_n = O_\omega(1)$ *reiškia, kad seka* (Z_n) *yra b.v. aprėžta.*

Taikymai

Savo darbe pademonstravome, kad 5 ir 6 teoremas tenkina tokie plačiai taikomi klasikiniai modeliai, kaip Ait-Sahalia:

$$X_t = x_0 + \int_0^t (a_1 X_s^{-1} - a_2 + a_3 X_s - a_4 X_s^r)\, ds + \sigma \int_0^t X_s^\beta \, dB_s^H \tag{8.16}$$

čia $x_0 > 0$, $r > 2\beta - 1$, $H \in (\frac{1}{2}, 1)$, $\beta > 1$, $\sigma > 0$, $a_1, a_2, a_3, a_4 > 0$; ar Hestono volatilumo modelis:

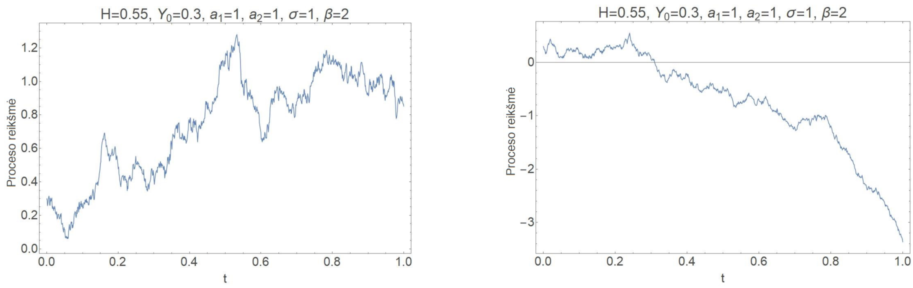
$$X_t = x_0 + \int_0^t a_1 X_s (a_2 - X_s)\, ds + \sigma \int_0^t X_s^\beta \, dB_s^H \tag{8.17}$$

su pradine reikšme $x_0 > 0$, čia $H \in (\frac{1}{2}, 1)$, $1 < \beta < 1.5$, deterministinės konstantos $a_1, a_2, \sigma > 0$.

Galiausiai kompiuteriniu modeliavimu parodėme, kad mūsų įrodymo schema nesuteikia jokių įžvalgų apie CKLS modelį:

$$X_t = x_0 + \int_0^t (a_1 - a_2 X_s)\, ds + \sigma \int_0^t X_s^\beta \, dB_s^H, \qquad \beta > 1, \quad H \in (\tfrac{1}{2}, 1), \tag{8.18}$$

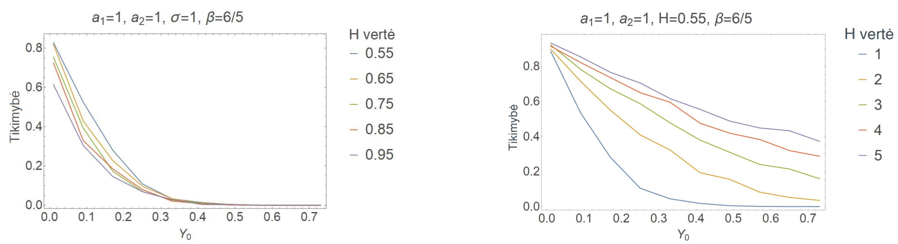
čia pradinė reikšmė $x_0 > 0$, o $a_1 > 0$, $a_2 \in \mathbb{R}, \sigma > 0$ – deterministinės konstantos. Modeliuodami pamatėme, kad Y gali įgyti arba neįgyti neigiamas reikšmes, kai $y_0 > 0$ (žr. 8.5 pav.).



(a) (b)

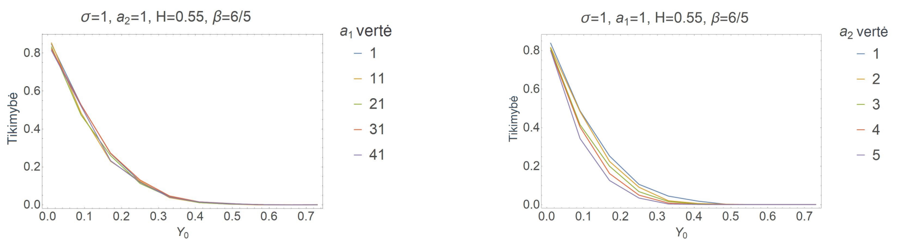
8.5 pav. Y trajektorijos: (**a**) su neigiamomis reikšmėmis. (**b**) su teigiamomis reikšmėmis.

Taip pat ištyrėme kokia tikimybė, kad šios trajektorijos gali įgyti teigiamas / neigiamas reikšmes skirtingoms CKLS parametrų reikšmėms.



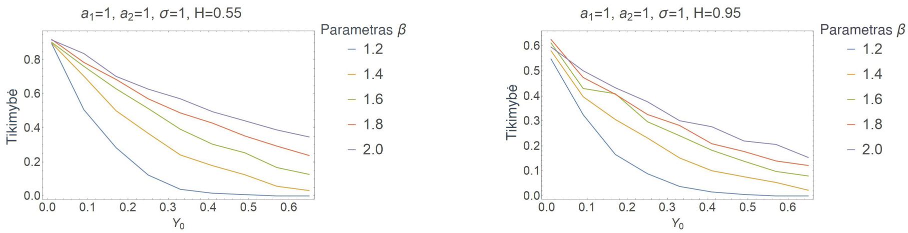
(a) (b)

8.6 pav. Neigiamų trajektorijos reikšmių tikimybė: (**a**) nuo pradinės reikšmės Y_0 skirtingiems H; (**b**) nuo pradinės Y_0 skirtingiems σ.



(a) (b)

8.7 pav. Neigiamų trajektorijos reikšmių tikimybė: (**a**) nuo a_1; (**b**) nuo a_2.



(a) (b)

8.8 pav. Neigiamų trajektorijos reikšmių tikimybė: (**a**) nuo β, kai $H = 0,55$; (**b**) nuo β, kai $H = 0,95$.

Nustatėme, kad (8.18) lygtis turi sprendinį $X_t = Y_t^{-1/(\beta-1)}$ iki tol, kol Y neįgyja nulinės reikšmės.

8.6.2 Patrajektoriui konverguojanti trupmeninės SDL aproksimacija

Tegul

$$X_t = x_0 + \int_0^t \alpha(X_s)\,ds + \int_0^t \sigma(X_s)\,dZ_s, \qquad x_0 \in \mathbb{R}, \quad t \in [0,T] \tag{8.19}$$

yra vienmatė SDL, valdoma kažkokio stochastinio proceso $Z = (Z_t)_{t \geqslant 0}$, $Z_0 = 0$, kurio trajektorijos $1/2 < \gamma < 1$ eilės Hiolderio trajektorijos. Šią lygtį vėl traktuosime kaip patrajektorę Riemann–Stieltjes integralinę lygtį, o kaip konkretų tokios lygties atvejį vėl galėsime nagrinėti SDL, kai $Z = B^H$.

Mūsų tikslas – sukonstruoti (8.19) SDL sprendinio aproksimaciją, įvedant tam tikras papildomas sąlygas difuzijos koeficientui.

Tegul kažkokiam $L > 0$ ir bet kokiems $x, y \in \mathbb{R}$,

$$|\alpha(x)| + |\sigma(x)| \leqslant L(1+|x|), \qquad |\sigma'(x)| \leqslant L, \tag{8.20}$$

$$|\alpha(x) - \alpha(y)| + |\sigma'(x) - \sigma'(y)| \leqslant L|x-y| \tag{8.21}$$

ir Z yra stochastinis procesas, kurio trajektorijos yra $1/2 < \gamma < 1$ eilės Hiolderio trajektorijos.

10 apibrėžimas. *Tegul* $1/2 < \gamma < 1$, $\alpha \in (1-\gamma, 1/2)$. *Tada* $W_\infty^\alpha([0,T])$ *žymėsime erdvę realiąsias vertes įgyjančių mačių funkcijų* $f\colon [0,T] \to \mathbb{R}$

tokių, kad

$$\|f\|_{\infty,\alpha;T} = \sup_{s\in[0,T]}\left(|f(s)| + \int_0^s |f(s)-f(u)|(s-u)^{-1-\alpha}du\right) < \infty.$$

Įrodoma [45] (žr. taip pat [48]), kad šias sąlygas tenkinanti (8.19) lygtis turi vienintelį sprendinį X tokį, kad $\|X\|_{\alpha,\infty,T} < \infty$ b.v. kiekvienam $\alpha \in (1-\gamma, \frac{1}{2})$.

Be sąlygų (8.20)-(8.21), taip pat pareikalausime, kad difuzijos koeficientas tenkintų sąlygą:

(**H**) $\inf_{x\in\mathbb{R}} \sigma(x) > 0$.

Pastebime, kad dėl (8.21) sąlygos, funkcija $1/\sigma(x)$ yra tolydžiai diferencijuojama virš $\mathbb{R}$. Taigi, iš (**H**) sąlygos išplaukia, kad Lamperti transformacija

$$F(x) = \int_0^x \frac{1}{\sigma(y)}\,dy, \qquad x \in \mathbb{R}$$

turi atvirkštinę funkciją $F^{-1} : \mathbb{R} \to \mathbb{R}$, kuri yra griežtai monotoniška ir diferencijuojama

$$(F^{-1})'(x) = \sigma(F^{-1}(x)), \qquad x \in \mathbb{R}. \tag{8.22}$$

Tegul $Y_t = F(X_t)$. Tada, pasinaudoję sudėtinės funkcijos integravimo formule, gausime

$$Y_t = Y_0 + \int_0^t F'(X_s)\,dX_s = Y_0 + \int_0^t \frac{\alpha(X_s)}{\sigma(X_s)}\,ds + Z_t = y_0 + \int_0^t f(Y_s)\,ds + Z_t,$$

čia $y_0 = F(x_0)$,

$$f(x) = \widehat{f}(F^{-1}(x)), \qquad \widehat{f}(x) = \frac{\alpha(x)}{\sigma(x)}.$$

Kadangi dėl (8.20)–(8.21) sąlygų, (8.19) lygtis turi vienintelį sprendinį, tai lygtis

$$Y_t = y_0 + \int_0^t f(Y_s)\,ds + Z_t \tag{8.23}$$

dėl tų pačių sąlygų taip pat turi vienintelį sprendinį.

Sąlygos

Mūsų pagrindiniams rezultatams įrodyti funkcijai f turi tenkinti šias papildomas sąlygas:

($\mathbf{C}_0$) f yra tolydžiai diferencijuojama $\mathbb{R}$;

($\mathbf{C}_1$) egzistuoja konstanta $K \in \mathbb{R}$ tokia, kad $f'(x) \leqslant K$ kiekvienam $x \in \mathbb{R}$;

($\mathbf{C}_2$) egzistuoja konstanta $M \geqslant 0$ tokia, kad $g'(x) \geqslant -M$, kiekvienam $x \in \mathbb{R}$, čia $g(x) = f(x)f'(x)$;

($\mathbf{C}_3$) egzistuoja du kartus diferencijuojama funkcija f virš $\mathbb{R}$ ir egzistuoja konstanta $N \geqslant 0$ tokia, kad $|f''(x)| \leqslant N$ kiekvienam $x \in \mathbb{R}$.

Teoremos

Tegul $\pi^n = \{t_k^n = \frac{k}{n}T, 1 \leqslant k \leqslant n\}$ yra tolygių intervalo $[0,T]$ skaidinių seka, o $h = t_k^n - t_{k-1}^n$, $1 \leqslant k \leqslant n$. (4.5) lygties sprendiniui Y apibrėšime atvirkštinę aproksimacinę schemą:

$$\begin{aligned} &Y_{n,k+1} - f(Y_{n,k+1})h + f'(Y_{n,k+1})f(Y_{n,k+1})\,\frac{h^2}{2} \\ &\quad = Y_{n,k} + (Z_{t_{k+1}^n} - Z_{t_k^n}) - f'(Y_{n,k}) \int_{t_k^n}^{t_{k+1}^n} (Z_{t_{k+1}^n} - Z_s)ds, \qquad \text{(A)} \\ &\quad Y_{n,0} = y_0,\ 0 \leqslant k \leqslant n-1. \end{aligned}$$

Tegul

$$h_0 := \frac{\sqrt{2M + (K^+)^2} - K^+}{M}, \qquad (8.24)$$

čia K ir M konstantos iš ($\mathbf{C}_1$) ir ($\mathbf{C}_2$) sąlygų, $K^+ = \max\{0, K\}$.

7 teorema. *Tegul (4.5) lygtyje esanti funkcija f tenkina sąlygas ($\boldsymbol{C}_0$)–($\boldsymbol{C}_3$). Tarkime, kad tolygių intervalo $[0,T]$ skaidinių π^n sekai $h < h_0$. Tada, jei $\gamma \in (\frac{1}{2}, 1)$ gausime, kad*

$$\max_{1 \leqslant k \leqslant n} |Y_{t_k^n} - Y_{n,k}| = O_\omega(h^{2\gamma}).$$

1 pastaba. *Pastebime, kad šis rezultatas nėra taikytinas CKLS, Heston-3/2 volatilumo ar Ait-Sahalia modeliams, nes jie netenkina ($\boldsymbol{C}_3$) sąlygos.*

8 teorema. *Tegul (8.19) SDL turi vienintelį sprendinį ir 10 teoremos* *sąlygos yra tenkinamos. Tada, kai* $\gamma \in (\frac{1}{2}, 1)$*, gausime, kad*

$$\max_{1 \leqslant k \leqslant n} |X_{t_k^n} - F^{-1}(Y_{n,k})| = O_\omega(h^{2\gamma}).$$

Taikymai

Mūsų gauti teoriniai rezultatai yra taikytini konkrečioms SDL. Pavyzdžiui, darbe pademonstravome tai Pearsono trupmeninio difuzijos procesui:

$$X_t = x_0 + \Phi(X_t) - \Phi(x_0) + \int_0^t \alpha(X_t)\,dt + \int_0^t \sigma(X_t)\,dB_t^H, \qquad t \geq 0, \tag{8.25}$$

su

$$\alpha(x) = b - ax, \qquad \sigma(x) = \sqrt{\sigma_0 + \sigma_1 x + \sigma_2 x^2}, \qquad \sigma_2 > 0; a, b \in \mathbb{R}$$

kai σ_i, $i = 0,1,2$ yra tokie, kad kvadratinė šaknis egzistuoja ir $\inf_{x \in \mathbb{R}} \sigma(x) > 0$, t. y., $\sigma_0 + \sigma_1 x + \sigma_2 x^2 > 0$.

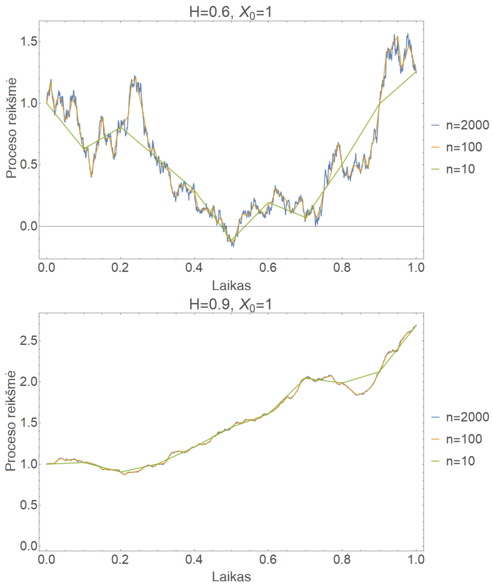
Paėmę konkrečias σ_i, $i = 0,1,2$ reikšmes

$$\sigma(x) = \sqrt{x^2 + 2x + 2}, \qquad a = 1, \qquad b = 2. \tag{8.26}$$

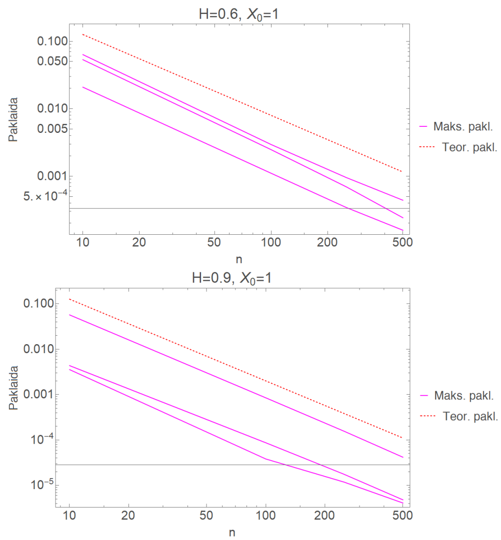
bei (8.13) ir Lamperti transformaciją apjungiančią schemą

$$\begin{aligned} F(X_{n,k+1}) - \widehat{f}(X_{n,k+1})h + \widehat{f}'(X_{n,k+1})\widehat{f}(X_{n,k+1})\,\frac{h^2}{2} \\ = F(X_{n,k}) + (B^H_{t^n_{k+1}} - B^H_{t^n_k}) - \widehat{f}'(X_{n,k}) \int_{t^n_k}^{t^n_{k+1}} (B^H_{t^n_{k+1}} - B^H_s)ds, \\ X_{n,0} = x_0,\ 0 \leqslant k \leqslant n-1, \end{aligned} \tag{A1}$$

darbe pateikėme kompiuterinio modeliavimo rezultatus:



8.9 pav. Pearsono modelio trajektorijų aproksimacijos (8.26) sąlygoms



8.10 pav. Keleto Pearsono proceso trajektorijų aproksimacijų (8.26) sąlygoms maksimali paklaida, lyginant su teorine paklaida

Integruoto tBj aproksimacija

Pastebime, kad (A1) schemoje turime Rymano prasme integruoto trupmeninio Brauno judesio dedamąją. Ji negali būti suskaičiuota tiksliai.

Todėl savo darbe pasiūlėme jos aproksimaciją ir įrodėme, kad

1 lema. *Tegul*

$$Y_{n,m,k,T} = \int_{t_m}^{t_{m+1}} B_s^H ds - \frac{T}{n^2}\sum_{k=1}^{n} B_{s_{k,m}}^H.$$

Tada, kiekvienam $\epsilon : H + 3/2 > \epsilon > 1/2$ *egzistuoja atsitiktinis dydis* η_ε : $\mathbb{E}|\eta_\varepsilon| < \infty$ *toks, kad*

$$|Y_{n,m,k,T}| \le \eta_\varepsilon \cdot n^{-H-3/2+\varepsilon} \qquad b.v.$$

kiekvienam $n \in \mathbb{N}$.

8.6.3 Trupmeninių stochastinių diferencialinių lygčių klasė su minkštąja sienele

Dauguma darbų, kuriuose Vynerio proceso valdomos klasikinės stochastinės diferencialinės lygtys bendrinamos į trupmeninio Brauno judesio valdomas lygtis, pagrįsti uždaviniais, su kuriais susiduriama naudojant SDL finansų srityje (opcionų kainodara, stochastinis volatilumas ar palūkanų normų modeliavimas). Tačiau stochastinių procesų taikymas gamtos moksluose (ypač biologijoje ir medicinoje) susilaukia daug mažesnio matematikų dėmesio, nors, anot Fulinskio [20], būtent šioje srityje atsirandantys uždaviniai „atveria visiškai naują skyrių Brauno judesio tyrimų istorijoje“. Pažvelgus atidžiau, paaiškėja, kad gamtos mokslai siūlo unikalių reikalavimų ir iššūkių taikant SDL.

Vienas iš iššūkių, susijęs su procesais, veikiančiais fizikinėje erdvėje (priešingai nei finansų srityje, kuri veikia nematerialioje, įsivaizduojamoje erdvėje), yra šios fizikinės aplinkos poveikis pačiam procesui. Pavyzdžiui, sienelės, ribojančios proceso trajektorijų sklaidą. Vojta ir kt. [55] pateikia visą tokių sienelių, kurias jie suklasifikuoja priklausomai nuo proceso diskrečiojo pavidalo, sąrašą.

Iš visų šių tipų, mus domina minkštoji sienelė, kurią apibrėžia rekursinė išraiška

$$x_{n+1} = x_n + \xi_n + G(x_n),$$

čia x_n – proceso reikšmė, ξ_n – diskretusis Gauso triukšmas, G – atstumianti jėga. Šis modelis yra mažai tyrinėtas bei rodo labai įdomų, subtilų elgesį, kuris atsiranda dėl atstūmimo jėgos vietoje atspindžio įvedimo. Priešingai nei standartinės SDL su atspindžiu, kurioje procesas negali peržengti kietosios sienelės, minkštoji sienelė yra atstumianti, tačiau nėra nepralaidi. Peržengęs minkštosios sienelės ribą, procesas patiria tam tikro dydžio jėgą priešinga kryptimi, o esant toli nuo sienelės, ši jėga veikia silpnai. Kadangi šis proceso trajektorijas slopinantis poveikis išvengia elastingumo arba neelastingumo prielaidų, toks matematinis modelis daugeliu atveju daug labiau atitinka realius gamtoje vykstančius procesus.

Taigi paimkime vienmatę SDL

$$Y_t = Y_0 + G(Y_t) - G(Y_0) + \int_0^t f(s, Y_s)\, ds + Z_t, \qquad t \in [0, T], \tag{8.27}$$

kurią valdo kažkoks stochastinis procesas $Z = (Z_t)_{t \geqslant 0}$, $Z_0 = 0$, su tolydžiomis trajektorijomis, čia $G : \mathbb{R} \to [0, \infty)$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ yra tolydžios funkcijos. Vėlgi kaip atskirą tokio proceso atvejį galime imti $Z = B^H$.

Tokią lygtį vadinsime – *SDL su minkštąja sienele*, funkcija G – atstumianti jėga, kurios didumas keičiasi atitinkamai nuo proceso padėties sienelės atžvilgiu. Pavyzdžiui, tokia atstumianti jėga su sienele taške w gali būti aprašoma eksponentine funkcija

$$G(x) = G_0 \exp\{-\lambda(x - w)\},$$

kurią charakterizuoja jėgos amplitudė G_0 ir gesimo konstanta λ (žr. [55]).

Nagrinėkime SDL, užrašomą pavidalu

$$Y_t = Y_0 + G(Y_t) - G(y_0) + \int_0^t f(s, Y_s)\, ds + Z_t, \qquad t \in [0, T], \tag{8.28}$$

valdomą tam tikro stochastinio proceso $Z = (Z_t)_{t \geqslant 0}, Z_0 = 0$, su tolydžiomis trajektorijomis.

Panaudojus funkciją $D(x) := x - G(x)$, (8.28) lygtį galima supaprastinti

$$D(Y_t) = D(Y_0) + \int_0^t f(s, Y_s)\, ds + Z_t. \tag{8.29}$$

Sąlygos

Norėdami įrodyti sprendinio egzistavimą ir vienatį įvesime keletą sąlygų:

(**A**) Kiekvienam $t \in [0,T]$ ir $x \in \mathbb{R}$ funkcijos f turi *augti tiesiškai*:

$$|f(t,x)| \leqslant K(1+|x|).$$

(**B**) Kiekvienam $t \in [0,T]$ ir $x \in \mathbb{R}$ funkcija f turi būti *tolydi Lipšico prasme*:

$$|f(t,x)-f(t,y)| \leqslant K|x-y|.$$

(**C**) *Tolydumas Hiolderio prasme laiko atžvilgiu*: egzistuoja $\beta \in (0,1]$ tokia, kad visiems $s,t \in [0,T]$ ir $x \in \mathbb{R}$

$$|f(s,x)-f(t,x)| \leqslant K|s-t|^{\beta}.$$

(**D**) Funkcija D yra siurjektyvi ir griežtai monotoniška.

(**E**) Egzistuoja konstanta $d > 0$, tokia kad

$$|D(x)-D(y)| \geqslant d|x-y|. \tag{8.30}$$

Teoremos

Apibrėžę šias sąlygas, galime suformuluoti (8.28) lygties sprendinio vienaties ir egzistavimo teoremą

9 teorema. *Tarkime, kad sąlygos (**A**), (**B**), (**D**) ir (**E**) yra tenkinamos, o $Z \in C([0,T])$, tai* (8.28) *lygtis turi vienintelį sprendinį $Y \in C([0,T])$. Jei stochastinis procesas $Z \in C^{\gamma}([0,T])$, tai $Y \in C^{\gamma}([0,T])$, $\gamma \in (0,1)$.*

Dabar, (8.28) lygčiai sukonstruosime neišreikštinę Oilerio schemą. Tegul $\pi^n = \{t_k^n = (k/n)T, 1 \le k \le n\}$ yra tolygių intervalo $[0,T]$ seka ir $h_n = t_k^n - t_{k-1}^n$, $1 \le k \le n$. Tada galime apibrėžti schemą:

$$D(Y_{n,k+1}) = D(Y_{n,k}) + f(t_k^n, Y_{n,k})h_n + (Z_{t_{k+1}^n} - Z_{t_k^n}), \qquad Y_{n,0} = Y_0. \tag{8.31}$$

10 teorema. *Tegul* (***A***) $-$ (***E***) *sąlygos yra tenkinamos, o Y yra* (8.28) *lygties sprendinys toks, kad $Y \in C^{\gamma}([0,T])$, $\gamma \in (0,1)$. Tada,*

$$\max_{1 \leqslant k \leqslant n} |Y_{t_k^n} - Y_{n,k}| = O_{\omega}(h_n^{\theta}),$$

čia $\theta = \beta \wedge \gamma$.

1 išvada. *Jeigu yra tenkinamos 10 teoremos sąlygos, tai*

$$\sup_{0 \le t \le T} |Y_t - Y_t^n| = O_\omega(h_n^\theta), \quad \gamma \in (0,1), \tag{8.32}$$

čia

$$Y_t^n = Y_{n,k} + \frac{t - t_k^n}{h_n}(Y_{n,k+1} - Y_{n,k}), \quad t \in (t_k^n, t_{k+1}^n],\ k = 0, \ldots, n-1.$$

Dabar, tegul

$$X_t = X_0 + \Phi(X_t) - \Phi(X_0) + \int_0^t \alpha(X_s)\, ds + \int_0^t \sigma(X_s)\, dZ_s, \qquad t \in [0,T], \tag{8.33}$$

čia Φ, α, σ yra tolydžios funkcijos, $Z \in C^\gamma([0,T])$ ir griežtai teigiamu difuzijos koeficientu, t. y., $\inf_{x \in \mathbb{R}} \sigma(x) > 0$. Kadangi, (8.33) lygtyje esantis integralas yra patrajektoriui Riemann–Stieltjes integralas, tai ir visa lygtis yra patrajektoriui Riemann–Stieltjes integralinė lygtis. Patrajektorinis Riemann–Stieltjes integralas yra apibrėžtas, kai $\sigma(X)$ yra tolydi Hiolderio prasme, turi eilę λ ir $\lambda + \gamma > 1$.

Įvedus papildomą sąlygą

(H) $\Phi : \mathbb{R} \to \mathbb{R}$ yra tolydžiai diferencijuojama funkcija ir egzistuoja konstantos $0 \leqslant c < 1$ tokios, kad $\Phi'(x) \leqslant c$ kiekvienam $x \in \mathbb{R}$;

galime įrodyti

11 teorema. *Tarkime, kad* $\alpha(x)$ *ir* $\sigma(x)$ *yra tolydžiai diferencijuojamos funkcijos ir* $\inf_{x \in \mathbb{R}} \sigma(x) > 0$. *Jei funkcija* $\Phi(x)$ *tenkina* ***(H)*** *sąlygą ir kažkokiai konstantai* $C > 0$, *ir bet kokiam* $x \in \mathbb{R}$,

$$|\alpha'(x)| \leqslant C, \qquad |\sigma'(x)| \leqslant C, \qquad \left|\frac{\alpha(x)}{\sigma(x)}\right| \leqslant C \tag{8.34}$$

tai (8.33) *SDL turi vienintelį sprendinį* $X \in C^\gamma([0,T])$, $1/2 < \gamma < 1$.

12 teorema. *Tegul 11 teoremos sąlygos yra tenkinamos. Tada*

$$\max_{1 \leqslant k \leqslant n} |X_{t_k^n} - X_{n,k}| = O_\omega(h_n^\gamma), \qquad \gamma \in (1/2, 1),$$

čia $X_{n,k} = F^{-1}(Y_{n,k})$, *o funkcija*

$$F(x) = \int_0^x \frac{1}{\sigma(y)}\,dy.$$

Atkreipsime skaitytojo dėmesį, kad nors iš pirmo žvilgsnio sąlyga **(E)** gali atrodyti neįprasta ir ribojanti (ypač kartu su sąlyga **(D)**), iš tikrųjų ją tenkina labai plati funkcijų klasė, kurios įdomiausius atskirus atvejus analizavome darbe.

Taikymai

Norėdami parodyti, kad mūsų teorinius rezultatus galima taikyti praktiškai, darbe pirmiausiai įrodome, kad sąlygas (**A**)–(**E**) tenkina trupmeninis Pearsono difuzijos procesas

$$X_t = x_0 + \Phi(X_t) - \Phi(x_0) + \int_0^t \alpha(X_t)\,dt + \int_0^t \sigma(X_t)\,dB_t^H, \qquad t \geq 0, \tag{8.35}$$

su

$$\alpha(x) = b - ax, \qquad \sigma(x) = \sqrt{\sigma_0 + \sigma_1 x + \sigma_2 x^2}, \qquad \sigma_2 > 0;\, a, b \in \mathbb{R},$$

kai σ_i, $i = 0, 1, 2$ yra tokie, kad kvadratinė šaknis yra apibrėžta,
o $\inf_{x \in \mathbb{R}} \sigma(x) > 0$, t. y. $\sigma_0 + \sigma_1 x + \sigma_2 x^2 > 0$.

Vėliau, kompiuteriniu modeliavimu ištiriame trupmeninį Vasiceko procesą su minkštąja sienele

$$X_t = x_0 + G(X_t) - G(x_0) + \int_0^t (\beta - \alpha X(s))\,ds + \sigma B^H(t), \tag{8.36}$$

čia $t \geq 0, \alpha, \beta \in \mathbb{R}$, $\sigma \geq 0$. Modeliuodami panaudojome du skirtingus sąlygas (**D**)–(**E**) tenkinančius atstumiančios jėgos profilius.

Viengubą profilį:

$$G(x) = \begin{cases} G_1(x) = k_1 e^{-\lambda(x-a_1)}, & \text{jei } x > a_1 \\ G_2(x) = k_2(a_1 - x) + G_1(a_1), & \text{jei } x \leq a_1, \end{cases} \tag{8.37}$$

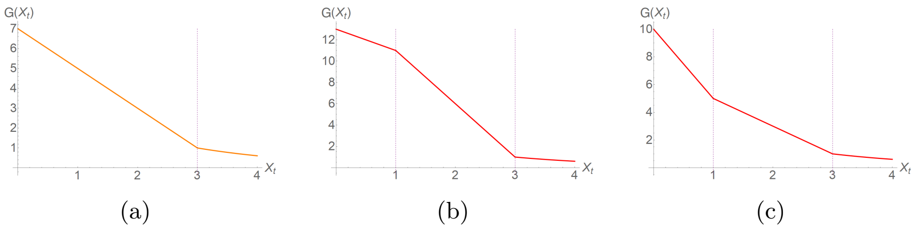
čia $\lambda \in (0,1)$; $a_1, k_1, k_2 > 0$.

Dvigubą profilį:

$$G(x) = \begin{cases} G_1(x) = k_1 e^{-\lambda(x-a_1)}, & \text{jei } x > a_1 \\ G_2(x) = k_2(a_1 - x) + G_1(a_1), & \text{jei } a_2 \leq x \leq a_1 \\ k_3(a_2 - x) + G_2(a_2), & \text{jei } x < a_2, \end{cases} \tag{8.38}$$

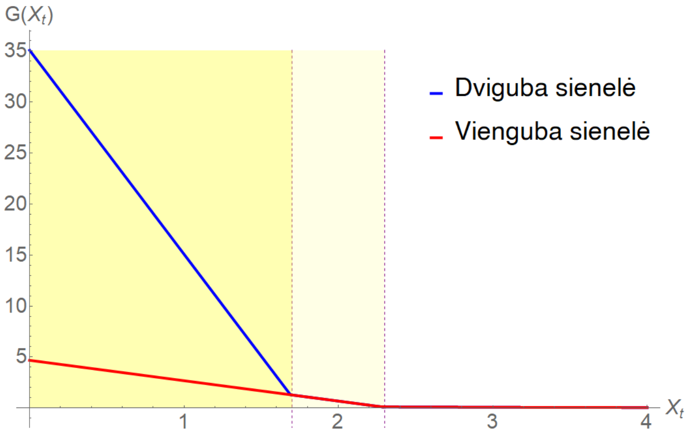
čia $\lambda \in (0,1)$; $a_1, a_2, k_1, k_2, k_3 > 0$.

Šie jėgos G profiliai pasirinkti modeliuoti, siekiant imituoti sparčius proceso aplinkos savybių pokyčius paprasčiausiomis priemonėmis (t. y. tiesine funkcija).



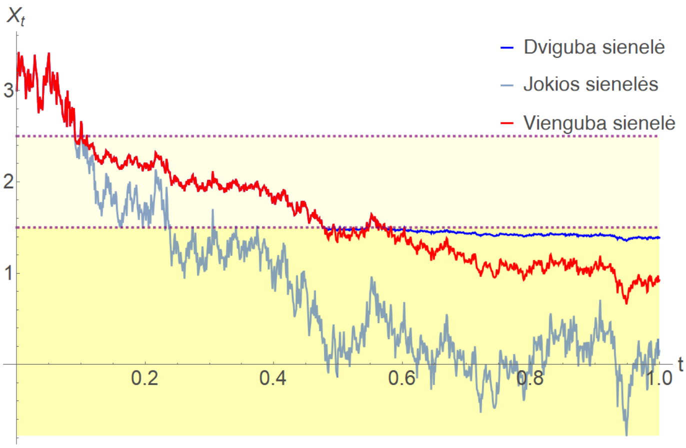
8.11 pav. Jėgos G profiliai. (**a**) Viengubas. (**b**) Dvigubas nuožulnėjantis. (**c**) Dvigubas statėjantis.

Panaudoję (8.31) schemą trupmeniniam Vasiceko procesui (8.36) ir jėgos profiliams (8.37) bei (8.38) gavome įvairaus elgesio šio proceso trajektorijų simuliacijas.



8.12 pav. Viengubas (8.37) ir dvigubas statėjantis (8.38) jėgos G profilis, kai $k_1 = 0,001, k_2 = 2,\ k_3 = 20, a_1 = 2,3, a_2 = 1,7$.

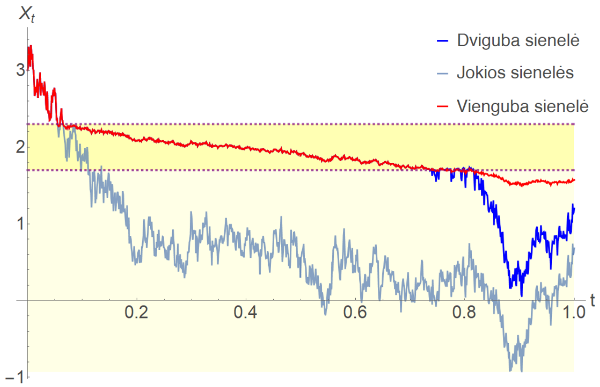
Kaip matome iš 8.13 pav., palyginus proceso trajektorijas su minkštosios sienelės ir be minkštosios sienelės jėgos poveikio, jėga G „stumia“ proceso trajektorijas link sienelės, o šio „stūmimo“ stiprumas priklauso nuo išvestinės G' dydžio (pakankamai didelės išvestinės turi beveik tiesinamąjį poveikį trajektorijoms).



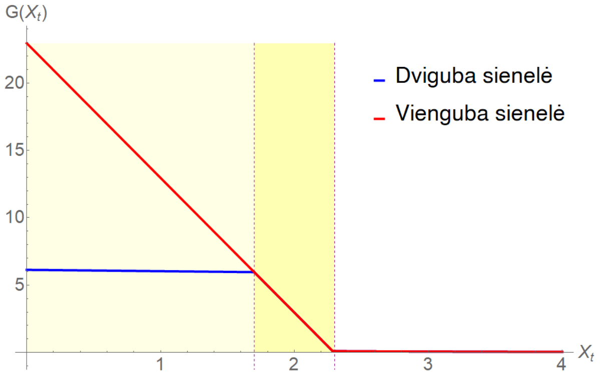
8.13 pav. Trupmeninio Vasiceko proceso trajektorijos, kai $\alpha = 1$, $\beta = 3, H = 0{,}3, \lambda = 0{,}5, T = 1, n = 1000$.

Dar įspūdingesnius reiškinius pamatysime stebėdami minkštųjų sienelių poveikį dvigubiems jėgos profiliams.

8.14 paveikslėlyje matome, kad trajektorijų volatilumas priklausomas ne nuo jėgos G dydžio, o tik nuo išvestinės G' didumo. Taigi, dvigubi jėgos profiliai (žr. 8.15 pav.) imituoja membraną tarp a_1 ir a_2, kurią „praduria“ trupmeninis Vasiceko procesas (žr. 8.14 pav.). Pastebime, kad palikusio intervalą $[a_2, a_1]$ proceso pokyčiai atstatomi ir tampa beveik identiški proceso be minkštosios sienelės poveikio pokyčiams.



8.14 pav. Trupmeninio Vasiceko proceso trajektorijos, kai $\alpha = 1, \beta = 5, H = 0,3, \lambda = 0,5, T = 1, n = 1000$.



8.15 pav. Viengubas (8.37) ir dvigubas nuožulnėjantis (8.38) jėgos G profilis, kai $k_1 = 0,001, k_2 = 10,\ k_3 = 0,1, a_1 = 2,3, a_2 = 1,7$.

8.7 Išvados

Apibendrinant disertacijoje nuveiktus darbus galima pateikti tokias išvadas:

- Mums pavyko suformuluoti pakankamai paprastas sąlygas (3.1) lygčiai, kurios užtikrina vienintelį teigiamą sprendinį. Be to, taikydami specifinę funkcijų kompozicijos integralo formulę ir Lamperti transformaciją, įrodėme, kad esant tam tikroms sąlygoms egzistuoja injekcija tarp (3.1) ir (3.2) SDL sprendinių.

- Parodėme, kad (3.1) ir (3.2) lygčių sprendiniai gali būti aproksimuoti neišreikštine Oilerio schema (3.5), išsaugant skaitinės schemos teigiamumą ir užtikrinant konvergavimą beveik visur.

- Pademonstravome, kad keletas klasikinių stochastinių modelių (t. y. Ait-Sahalia, Heston) tenkina mūsų teoremų sąlygas (3.2) lygčiai ir gali būti aproksimuoti neišreikštine Oilerio schema (3.5). Be to, aptarėme savo metodikos trūkumus, taikant ją CKLS modeliui.

- Pasiūlėme Milšteino tipo aproksimacinę schemą (4.1) SDL ir parodėme jos aukštą konvergavimo greitį. Dar daugiau, parodėme, kad ši aproksimacinė schema gali būti pritaikyta Pearsono difuzijos modeliui.

- Apibendrinę tam tikrus rezultatus iš tikimybių ir skaičių teorijos, pasiūlėme integruoto trupmeninio Brauno judesio aproksimacinę schemą su geru konvergavimo greičiu.

- Suformulavome stochastiniu požiūriu matematiškai griežtą būdą apibrėžti procesą su minkštąja sienele trupmeninės stochastinės diferencialinės lygties pavidalu. Tada pasiūlėme sąlygas ir įrodėme, kad jas tenkinant tokia tSDL turi vienintelį sprendinį. Parodėme, kad šis sąlygų rinkinys apima plačią funkcijų ir procesų klasę.

- Pasiūlėme ir ištyrėme proceso su minkštąja sienele aproksimacinę schemą, kuri turi gerą konvergavimo greitį bei yra lengvai panaudojama proceso trajektorijoms generuoti. Šis teiginys ištyrinėtas konkrečiu Vasiceko proceso atveju skirtingo sudėtingumo sienelėms.

- Sukonstravome stiprų ir asimptotiškai normalų Hursto parametro įvertį (3.2) ir (6.1) lygčių sprendiniams.

Appendix A

Several results on fBm

Recall that fBm $B^H = \{B_t^H, t \geqslant 0\}$ with the Hurst index $H \in (0,1)$ is a real-valued continuous centered Gaussian process with covariance

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}).$$

For $H = \frac{1}{2}$, fBm is a Brownian motion. To consider the strong consistency and asymptotic normality of the given estimators, we need several facts regarding B^H.

Limit results (see [33], [38]). Let

$$V_{n,T}^{(2)\widehat{B}^H} = \frac{n^{2H-1}}{T^{2H}(4-2^{2H})} \sum_{k=1}^{n-1} (\Delta_{n,k}^{(2)} B^H)^2, \qquad H \neq \frac{1}{2}.$$

Then (see [38], pp. 46, 52, 58, 66)

$$V_{n,T}^{(2)\widehat{B}^H} \xrightarrow[n\to\infty]{} 1 \quad \text{a.s.}$$

and

$$\sqrt{n} \begin{pmatrix} V_{n,T}^{(2)\widehat{B}^H} - 1 \\ V_{2n,T}^{(2)\widehat{B}^H} - 1 \end{pmatrix} \xrightarrow{d} \mathcal{N}(0; \Sigma_H), \qquad \Sigma_H = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix}, \quad \text{(A.1)}$$

where $\mathcal{N}(0;\Sigma_H)$ is a Gaussian vector with

$$\Sigma_{11} = 2\left(1+\frac{2}{(4-2^{2H})^2}\sum_{j=1}^{\infty}\widehat{\rho}_H^2(j)\right), \qquad \Sigma_{22} = \frac{1}{2}\Sigma_{11},$$

$$\Sigma_{12} = \Sigma_{21} = \frac{1}{2^{2H}(4-2^{2H})^2}\sum_{j\in\mathbb{Z}}\widetilde{\rho}_H^2(j),$$

$$\widehat{\rho}_H(j) = \frac{1}{2}\left[-6|j|^{2H}-|j-2|^{2H}-|j+2|^{2H}+4|j-1|^{2H}+4|j+1|^{2H}\right],$$

$$\widetilde{\rho}_H(j) = \frac{1}{2}\left[|j+1|^{2H}+2|j+2|^{2H}-|j+3|^{2H}\right] + \frac{1}{2}\left[|j-1|^{2H}-4|j|^{2H}-|j-3|^{2H}+2|j-2|^{2H}\right].$$

Moreover,

$$V_{n,T}^{(2)\widehat{B}^H} = 1 + O_\omega\left(n^{-1/2}\ln^{1/2} n\right) \tag{A.2}$$

and

$$\sqrt{n}\ln\frac{V_{2n,T}^{\widehat{B}^H}}{V_{n,T}^{\widehat{B}^H}} \xrightarrow{d} \mathcal{N}(0,\sigma_H^2)$$

with $\sigma_H^2 = \frac{3}{2}\Sigma_{11} - 2\Sigma_{12}$.

Appendix B

Supplementary results

Lemma 6 (Discrete version of Gronwall's lemma [23])**.** *Let* (y_n) *and* (g_n) *be non-negative sequences and* c *a non-negative constant. If*

$$y_n \leqslant c + \sum_{j=0}^{n-1} g_j y_j \qquad \textit{for } n \geqslant 0,$$

then

$$y_n \leqslant c \prod_{0 \leqslant j < n} (1 + g_j) \leqslant c \exp\left(\sum_{0 \leqslant j < n} g_j \right) \qquad \textit{for } n \geqslant 0.$$

Publications by the author

- K. Kubilius, A. Medžiūnas, *A Class of Fractional Stochastic Differential Equations with a Soft Wall*, Fractal Fract. 7(2) (2023).
- K. Kubilius, A. Medžiūnas, *Pathwise Convergent Approximation for the Fractional SDEs*, Mathematics 10(4) (2022).
- K. Kubilius, A. Medžiūnas, *Positive Solutions of the Fractional SDEs with Non-Lipschitz Diffusion Coefficient*, Mathematics 9(1) (2021).
- A. Medžiūnas, *On the Congruence of Finite Generalized Harmonic Numbers Sums Modulo p^2*, Ann. Pol. Math. 126(3) (2020) 279-292.

Curriculum Vitae (Aidas Medžiūnas)

Employment History

- 2023 - .... **Lecturer**, Faculty of Fundamental Sciences, VILNIUS TECH University.
- 2018 - .... **Junior Assistant**, Institute of Applied Mathematics, Vilnius University.
- 2016 - 2018 **Mathematics and Informatics Teacher**, Karalienė Morta School.

Education

- 2018 - 2023 **Ph.D., Vilnius University** in Mathematics. Thesis title: *Various Classes of Stochastic Differential Equations: Existence, Uniqueness, and Approximation.*
- 2016 - 2018 **M.Sc., Vilnius University** in Mathematics. Thesis title: *Hurst index estimates for the Langevin equation with mixed fractional Brownian motion.*
- 2012 - 2016 **B.Sc., Vilnius University** in Mathematics. Thesis title: *Bolus design for electron beam radiotherapy cancer treatment planning.*

Languages

- English – C2.
- Russian – C1.
- Lithuanian – native.

Vilniaus universiteto leidykla
Saulėtekio al. 9, III rūmai, LT-10222 Vilnius
El. p. info@leidykla.vu.lt, www.leidykla.vu.lt
bookshop.vu.lt, journals.vu.lt
Tiražas 15 egz.